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Simple Geometry without Coordinates

Building an Exotic Plane with Model Theory

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Abstract:

When we are first taught to draw a plane, we are taught to start by drawing two straight lines with equidistant numbers that serve as coordinates. These coordinates tend to come from a division ring, and they represent a glimpse of the connection between geometry and algebra. There are planes that do not admit coordinates over a division ring, but constructing them is complicated. Is it however possible to construct a simple plane without coordinates?

In the 1980s, Boris Zilber faced a similar question while studying the geometry of sets, and he conjectured that it was not possible. He believed that the geometry of strongly minimal sets must either be trivial, modular, or field-like. His conjecture became known as Zilber's trichotomy, and it was disproved ten years later by Ehud Hrushovski, who constructed a strongly minimal set whose geometry was too rich to be modular and too sparse to be field-like. Hrushovski's construction shook model theory, and many mathematicians used it to build astonishing structures. One of these mathematicians was John Baldwin, who used Hrushovski's techniques to build an almost strongly minimal, non-Desarguesian projective plane. A model is almost strongly minimal when all of its elements are in the algebraic closure of a strongly minimal set, and a projective plane is non-Desarguesian when it does not satisfy Desargues' theorem, which implies that it does not admit coordinates over a division ring. Baldwin's plane is then a simple projective plane without coordinates.

The goal of this thesis is to construct this plane from the ground up, providing all the details Baldwin skipped, so that any reader with a basic understanding of mathematical logic can follow it. We will construct the plane as an infinite graph. We will start by defining the class of finite square-free graphs K, from which we will remove redundant copies of certain graphs to obtain K^*. We will then show that K^* satisfies the amalgamation property, allowing us to combine its graphs into a single, infinite, K^*-universal-homogeneous graph M, the Fraïssé limit of the class K^*. This graph is Baldwin's plane, so we will show that it is an almost strongly minimal model, and a non-Desarguesian projective plane.

Keywords: Model theory, geometric model theory, stability theory, Fraïssé limit, Hrushovski construction, non-Desarguesian projective plane.

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Introduction

> In that Empire, the Art of Cartography attained such Perfection that the map of a single Province occupied the entirety of a City, and the map of the Empire, the entirety of a Province. In time, those Unconscionable Maps no longer satisfied, and the Cartographers Guilds struck a Map of the Empire whose size was that of the Empire, and which coincided point for point with it.
>
> — Jorge Luis Borges

The natural sciences study intricate structures. Physicists study matter and energy, chemists study atoms and molecules, and biologists study living organisms. To study these intricate structures, scientists develop models that capture their behavior. For these models to be useful, they must be both accurate and simple. Borges' cartographers let their model grow until it matched reality point for point, and in doing so made it useless. But what does it mean for a model to be simple?

Based on Borges' analogy, we could say that a model is simple if it is small, like a map that you can hold in your hands, but we could also say that a model is simple if it has a simple structure, like a map with a coordinate system. So what definition should we choose? Are they equivalent? Is it possible to have a small map whose structure is so complex that you cannot define a coordinate system?

To answer this question, we turn to model theory, a branch of mathematics that studies the structures of models. The first step to answering our question, however, is to formalize it. We will say that a model is *small* when its elements are easy to define. We can define the elements of a model using first-order formulas with parameters, so we can say that a model is small when the set of parameters we need to define its elements is small. But what is a small set of parameters? Requiring it to be finite would be too restrictive, so model theorists usually settle for *strongly minimal* sets, infinite sets that behave like finite sets in terms of definability. A model is then small when we can define its elements using first-order formulas with parameters over a strongly minimal set.

Now that we have formalized what it means for a model to be small, we need to formalize what it means for its structure to be simple. We will refer to the structure of a model as its geometry, and we will say that the geometry of a model is simple when we can define a sensible coordinate system to identify its points. By a *sensible* coordinate system we mean a coordinate system over a division ring, where the coordinates are structured and not just spread around randomly. The geometry of a model is then simple when we can define a coordinate system over a division ring.

We can now rephrase our original question and, instead of asking "Can small maps have complex structures?", we ask "Is there a model definable over a strongly minimal set that does not have a coordinate system over a division ring?"

In 1984, Boris Zilber asked himself this question and conjectured the intuitive answer. No. Simple models always have simple geometries, so the geometry of any model that is definable over a strongly minimal set always admits a coordinate system over a division ring. Zilber even went a step further, and conjectured that the geometry of any strongly minimal set is either trivial (you can only draw points), modular (you can only draw points and lines), or field-like (you can only draw points, lines, and curves) [6]. In 1993, however, Ehud Hrushovski refuted Zilber's conjecture by constructing a strongly minimal set whose geometry was too rich to be modular and too sparse to be field-like [3]. This construction reshaped model theory, and many mathematicians used it to build seemingly impossible structures. John Baldwin was one of them, and he adapted Hrushovski's construction to build a projective plane that was definable over a strongly minimal set, but admitted no coordinate system over a division ring. The mathematical equivalent of a small map with a complex structure.

My goal in this thesis is to construct this model from the ground up. Baldwin constructed it in a research paper, so he skipped many of the details that make the proofs work. My goal is then to give all the details and make the proofs so clear that any reader with a background in mathematical logic can follow them. To construct the model, we will start by defining the class of finite square-free graphs K. Then we will collapse K by restricting certain graphs to obtain the subclass K^*, for which we will show that we can build a K^*-universal-homogeneous graph M. We will then show that we can define the elements of M over a strongly minimal set, but that the geometry of M forms a projective plane that does not admit a coordinate system over a division ring, showing then that M is a small map with a complex structure.

Preliminaries

Chapters 1 and 2 do not have any prerequisites other than a basic understanding of mathematical proofs, but Chapters 3 and 4 expect readers to be familiar with mathematical logic as taught in most introductory courses. In particular, both chapters expect readers to know what first-order structures, theories, and formulas are, and to be familiar with compactness, invariance under isomorphisms and automorphisms, and the Löwenheim-Skolem theorems. As a concrete reference, I will assume throughout the thesis that readers are familiar with Sections 2–4 of the lecture notes for the course Mathematical Logic by Jouko Väänänen [5].

We will also simplify certain notation to improve readability. Throughout this thesis, we work exclusively in first-order logic, so we will omit the qualifier "first-order" when referring to notions such as formulas. Similarly, when the language L is clear from the context, we will omit it from the notation and write *structure*, *theory*, and *formula* instead of L-structure, L-theory, and L-formula. When the model M is clear from the context, we also omit it from subscripts and superscripts, writing, for example, $\overline{A}$ instead of $\overline{A}^M$, ℓ_b instead of ℓ_b^M, and $\mathrm{acl}(X)$ instead of $\mathrm{acl}_M(X)$.

To simplify the notation even further, we will omit the symbol $\cup$, writing, for example, AB instead of $A \cup B$. We also define, for all structures M and formulas $\varphi(x, a)$,

$$\varphi(M, a) := \{\, x \in M \mid M \models \varphi(x, a) \,\},$$

and for every $n \in \mathbb{N}$,

$$\exists^n x\, \varphi(x) := \exists x_1 \ldots \exists x_n \left(\bigwedge_{1 \leq i \leq n} \varphi(x_i) \wedge \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right).$$

Chapter 1

The Class of Finite Square-Free Graphs

The first step to begin our construction is to set up the playing field. Since we want to define a projective plane, a natural starting place would be a geometric object like a manifold. But manifolds are complicated objects and their nuances would distract us from our goal. We need a simple object we can understand and control, so that we can forget about the object itself and focus on its model-theoric and geometric structure. But what could such an object be? There are different options, but one of the simplest is a graph. Graphs are discrete, easy to visualize, and can have incredibly complicated structures, so they constitute the perfect playing field. Rich enough to exhibit the properties we want, but simple enough to avoid inherent complications.

Definition 1.1. A *graph* is a tuple (V, R) where V is a set and R is a symmetric and irreflexive relation.

For us, then, graphs are undirected, do not contain loops, and have at most a single edge between any two vertices. Such graphs are usually called *simple graphs*, but we will simply refer to them as graphs.

The goal of this thesis is to define a "simple" model, so to avoid making the same mistake as Borges' cartographers, we will start by defining a way to measure the complexity of a graph. In particular, we will measure the complexity of a graph by computing the relation between its number of vertices and edges. Every vertex adds degrees of freedom to the graph, so vertices increase complexity, while every edge relates two vertices together, so edges reduce complexity. Thus, a graph with many vertices and few edges is complex, while a graph with many edges and few vertices is simple.

Definition 1.2. Let A be a finite graph. The *predimension* of A is

$$\delta(A) = 2\,|A| - e(A),$$

where $|A|$ is the number of vertices of A and $e(A)$ is the number of edges.

The predimension function quantifies the complexity of graphs, and it is the only piece of mathematical machinery we need to define the class of graphs we will work with. We only impose two restrictions on the graphs of our class: they cannot be too simple, so there is a lower bound on their predimension, and they cannot contain squares, so any two vertices have at most one common neighbor.

Definition 1.3. Let K be the collection of all finite graphs A that do not contain embedded squares and satisfy $\delta(X) \geq 2$ for every non-empty $X \subseteq A$.

Remark 1.4. $\delta(\emptyset) = 0$, so $\emptyset \notin K$.

1.1 Relative Predimension

The graphs in K constitute the building blocks of the graph we are after, but to keep the complexity of our graphs under control, we need to be able to talk about how the complexities of different graphs relate to each other. In particular, we would like to know whether attaching a graph B to a graph A increases or decreases the complexity of A.

We will work with unions of graphs throughout most of the thesis, so to simplify the notation, for any two graphs A and B, we write AB instead of $A \cup B$.

Definition 1.5. Let A and B be two finite graphs. The *relative predimension* of B over A, denoted $\delta(B/A)$, is given by the function

$$\delta(B/A) = \delta(BA) - \delta(A).$$

We say that the graph A is the *base* and the graph B is the *extension*.

Remark 1.6. For any graph B, $\delta(B) = \delta(B/\emptyset)$ and $\delta(B/B) = 0$.

The relative predimension mathematically quantifies the complexity an extension adds to a graph. Intuitively, an extension simplifies the base when the relative predimension is negative, and complicates it when the relative predimension is positive. To get a better understanding of the relative predimension, we introduce some basic results.

Lemma 1.7. $\delta(B/A) = \delta(B - A/A) = \delta(BA/A)$.

Proof.

$$\begin{aligned}\delta(B/A) &= \delta(BA) - \delta(A) \\ &= \delta((B - A)A) - \delta(A) = \delta(B - A/A) \\ &= \delta(BAA) - \delta(A) = \delta(BA/A).\end{aligned}$$

□

The edges of the extension that connect the extension to the base give us the key to understanding the relative predimension. If an extension is disjoint from the base, for example, the relative predimension is simply the predimension of the extension minus the number of connecting edges.

Definition 1.8. Let A and B be two finite graphs. We write $r(A, B)$ for the number of edges with one vertex in A and one in B.

Proposition 1.9. *If $A \cap B = \emptyset$, then $\delta(A/B) = \delta(A) - r(A, B)$.*

Proof.

$$\begin{aligned}\delta(A/B) &= \delta(AB) - \delta(B) \\ &= 2\,|AB| - e(AB) - (2\,|B| - e(B)) \\ &= 2\,|A| + 2\,|B| - e(A) - e(B) - r(A, B) - 2\,|B| + e(B) \\ &= 2\,|A| - e(A) - r(A, B) \\ &= \delta(A) - r(A, B).\end{aligned}$$

□

This reduction of the relative predimension to the predimension has two corollaries that we will use again and again.

Corollary 1.10. *If $A \subseteq B$ and $C \cap B = \emptyset$, then*

$$\delta(C/A) = \delta(C/B) \iff r(C, B - A) = 0.$$

Proof. By Proposition 1.9,

$$\begin{aligned}\delta(C/A) = \delta(C/B) &\iff \delta(C) - r(C, A) = \delta(C) - r(C, B) \\ &\iff r(C, A) = r(C, B) \\ &\iff r(C, A) = r(C, A) + r(C, B - A) \\ &\iff r(C, B - A) = 0.\end{aligned}$$

□

Corollary 1.11. *If $A \subseteq B$ and $C \cap B \subseteq A$, then*

$$\delta(C/B) \leq \delta(C/A).$$

Proof. Notice that $C - B = C - A$, so by Lemma 1.7 and Proposition 1.9,

$$\begin{aligned}\delta(C/B) &= \delta(C - B/B) \\ &= \delta(C - B) - r(C - B, B) \\ &\leq \delta(C - B) - r(C - B, A) \\ &= \delta(C - B/A) \\ &= \delta(C - A/A) \\ &= \delta(C/A).\end{aligned}$$

□

To finish the section, we show three properties that allow us to reduce the relative predimension of a graph to the relative predimensions of its components.

Proposition 1.12 (Additivity)**.** *If* $AB \cap C = \emptyset$*, then*

$$\delta(C/AB) = \delta(C/A) + \delta(C/B) - \delta(C/A \cap B).$$

Proof. By Proposition 1.9,

$$\begin{aligned}\delta(C/AB) &= \delta(C) - r(C, AB)\\ &= \delta(C) - r(C, A) - r(C, B) + r(C, A \cap B)\\ &= \delta(C/A) - r(C, B) + r(C, A \cap B) + \delta(C) - \delta(C)\\ &= \delta(C/A) + \delta(C/B) - \delta(C/A \cap B).\end{aligned}$$

□

Proposition 1.13 (Predimension of unions)**.**

$$\delta(AB/C) = \delta(A/C) + \delta(B/AC),$$

Proof.

$$\begin{aligned}\delta(AB/C) &= \delta(ABC) - \delta(C)\\ &= \delta(ABC) - \delta(AC) + \delta(AC) - \delta(C)\\ &= \delta(B/AC) + \delta(A/C).\end{aligned}$$

□

Corollary 1.14 (Subadditivity)**.** *If* $A \cap B \subseteq C$*, then*

$$\delta(AB/C) \leq \delta(A/C) + \delta(B/C).$$

Proof. By Proposition 1.13 and Corollary 1.11,

$$\delta(AB/C) = \delta(B/AC) + \delta(A/C) \leq \delta(B/C) + \delta(A/C).$$

□

1.2 Strong Submodels

The relative predimension allows us to define mathematically what does it mean for a graph to be simple. Intuitively, a graph A is *simple* with respect to a graph B if $A \subseteq B$ and no subgraph of B reduces the complexity of A. In model theory, however, instead of saying that A is simple with respect to B, we say that A is a strong submodel of B.

Definition 1.15. A finite graph A is a *strong submodel* of a (possibly infinite) graph B, denoted $A \leq B$, if $A \subseteq B$ and $\delta(X/A) \geq 0$ for every finite $X \subseteq B$.

The definition of strong submodels applies to infinite graphs. We will not work with infinite graphs until Chapter 3, but it is important to consider them now since they will be essential in the future. After all, the simple plane without coordinates we are after is an infinite graph.

Remark 1.16. For every $b \in B$, $b \leq B$.

We now show two properties of the strong submodel relation.

Lemma 1.17 (Transitivity)**.** *If $A \leq B$ and $B \leq C$, then $A \leq C$.*

Proof. Let $X \subseteq C$ be a finite graph. Denote $X_B = X \cap B$ and $X_C = X \setminus B$. By assumption, $\delta(X_B/A) \geq 0$ and $\delta(X_C/B) \geq 0$, so by Proposition 1.13 and Corollary 1.11,

$$\delta(X/A) = \delta(X_B/A) + \delta(X_C/X_BA) \geq \delta(X_B/A) + \delta(X_C/B) \geq 0.$$

□

Lemma 1.18. *If $B \leq M$, $C \subseteq M$, and $\delta(C/B) = 0$, then $CB \leq M$.*

Proof. Let $X \subseteq M$. By Proposition 1.13,

$$\delta(X/CB) = \delta(XC/B) - \delta(C/B) = \delta(XC/B) \geq 0.$$

□

Our definition of the strong submodel relation involves all possible subgraphs of B, so it can be difficult to work with. To simplify our calculations, we show that the strong submodel relation is a local property.

Lemma 1.19 (Locality)**.** *If $A \leq B$ and $C \subseteq B$, then $A \cap C \leq C$.*

Proof. Let $X \subseteq C$ be a finite graph. By assumption, $A \leq B$ and $C \subseteq B$, so $\delta(X/A) \geq 0$, and by Corollary 1.11,

$$0 \leq \delta(X/A) \leq \delta(X/A \cap C).$$

□

The strong submodel relation is enough most of the time, but sometimes it is too restrictive, so we introduce relative strong submodels. The idea is that, if we have three graphs A, B, C such that $A \subseteq B$ and $A \leq C$, we may want to say that B preserves the relation between A and C even if $C \nsubseteq B$.

Definition 1.20. A graph A is a *C-strong submodel* of a (possibly infinite) graph B, denoted $A \leq_C B$, if $A \subseteq B$, $A \leq C$, and every $Y \subseteq B$ isomorphic to some $X \subseteq C$ satisfies $Y \cap A \leq Y$.

The relative strong submodel relation is a weakening of the strong submodel relation, as the following lemma shows.

Lemma 1.21. *If $A \leq B$, then $A \leq_C B$ for any graph C.*

Proof. The result follows from Lemma 1.19. □

1.3 Self-Sufficient Closure

The strong submodel relation therefore captures the idea of a graph that is simple with respect to another graph. But what happens when a graph A is not simple with respect to a graph B? That is, what happens if $A \subseteq B$ but $A \not\leq B$? The structure of A inside B may be incredibly complicated, so instead of trying to talk about A, we find an extension of A in B that simplifies A with respect to B.

Definition 1.22. A graph A is *closed* in a (possibly infinite) graph B if

$$\delta(A) = \inf\{\delta(X) \mid A \subseteq X \subseteq B \text{ and } X \text{ is finite}\}.$$

The self-sufficient closure of A in B, denoted $\overline{A}^B$, is a subset minimal, closed graph that contains A. When the ambient graph B is clear from the context, we simply write $\overline{A}$.

After the definition, there are three immediate questions we need to address. For any two graphs $A \subseteq B$, does $\overline{A}^B$ always exist? Is it unique? And is $\overline{A}^B \leq B$? The answers are yes, yes, and yes!

Proposition 1.23. *$A \subseteq B$ is closed if and only if $A \leq B$.*

Proof. If A is closed, then $\delta(A) = d_B(A)$, so $\delta(X/A) \geq 0$ for any $X \subseteq B$ and $A \leq B$. If A is not closed, then $\delta(A) > d_B(A)$, so there is a finite $X \subseteq B$ such that $\delta(X/A) < 0$ and $A \not\leq B$. □

Lemma 1.24. *For any graph $A \in K$ and any (possibly infinite) graph B, the graph $\overline{A}^B$ exists and is unique.*

Proof. The predimension of every $A \in K$ is $\delta(A) \in \mathbb{N} \setminus \{0, 1\}$, so there must be a graph in B that contains A and has minimal predimension. Thus, $\overline{A}^B$ exists. Suppose now towards contradiction that there are two distinct, subset minimal, closed sets $\overline{A}$ and $\overline{A}'$ containing A. By Proposition 1.23, $\overline{A} \leq B$, so by Lemma 1.19, $\overline{A} \cap \overline{A}' \leq \overline{A}'$. But then $\delta(\overline{A}'/\overline{A} \cap \overline{A}') \geq 0$ — contradicting either the closure or the minimality of $\overline{A}'$. □

The following property will allow us to simplify future notation, but its converse is not true. There are elements in $\overline{A}$ that increase the predimension of A even though they decrease the predimension of $\overline{A}$ overall, so if $\delta(b/A) \geq 0$, it does not follow that $b \notin \overline{A}$.

Lemma 1.25. *If $\delta(b/A) < 0$, then $b \in \overline{A}$.*

Proof. Suppose towards contradiction that there is $b \notin \overline{A}$ such that $\delta(b/A) < 0$. By Proposition 1.9 and Corollary 1.11,

$$\delta(b) - r(b, \overline{A}) = \delta(b/\overline{A}) \leq \delta(b/A) < 0.$$

But $\delta(\overline{A}b) = \delta(\overline{A}) + \delta(b) - r(b, \overline{A})$, so adding $\delta(\overline{A})$ to the above gives

$$\delta(\overline{A}b) = \delta(\overline{A}) + \delta(b) - r(b, \overline{A}) < \delta(\overline{A})$$

— a contradiction since $\overline{A}$ is closed. □

We now show three properties of the closure that will come in handy in the future.

Proposition 1.26 (Monotonicity)**.** *If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

Proof. The graph $\overline{B}$ is a closed graph containing A, and since by Lemma 1.24 the closure of a graph is unique, then $\overline{A} \subseteq \overline{B}$. □

Corollary 1.27. *If A and B are closed, then $A \cap B$ is closed.*

Proof. By the definition of closure, $A \cap B \subseteq \overline{A \cap B}$, and by Proposition 1.26, $\overline{A \cap B} \subseteq \overline{A} = A$ and $\overline{A \cap B} \subseteq \overline{B} = B$, so $\overline{A \cap B} \subseteq A \cap B$. □

Corollary 1.28. *If A and B are finite, then $\overline{AB} = \overline{A\overline{B}}$.*

Proof. By Proposition 1.26, $\overline{A}\,\overline{B} \subseteq \overline{AB} \subseteq \overline{A\overline{B}}$, and since $A\overline{B} \subseteq \overline{A}\,\overline{B}$, then by Proposition 1.26, $\overline{A\overline{B}} \subseteq \overline{\overline{AB}} = \overline{AB}$, so $\overline{AB} = \overline{X\overline{B}}$. □

1.4 Induced Dimension

Let $A \subseteq M \in K$ be graphs. We just showed that A has a unique closure $\overline{A}$ in M and that, if $A \not\leq M$, then A needs some extra vertices and edges to be a strong submodel of M. These extra vertices and edges do not add any new information to A. They are completely determined by A, so even though they are not part of A, it is as if they were. They are a “hidden” part of A. Measuring the complexity of A with the predimension, then, is a bit misleading. The predimension measures the complexity of A, but ignores its “hidden” part. It is a pedantic notion of complexity. So if we want to measure the *real* complexity of A, we need to measure the complexity of both the seen and hidden parts of A — the complexity of $\overline{A}$.

Definition 1.29. Let A be a finite graph and let M be a (possibly infinite) graph. The *dimension* of A in M is

$$d_M(A) = \delta(\overline{A}).$$

Again, as in the case of the predimension, to keep the complexity of our graphs under control, we need to be able to measure how the dimensions of different graphs relate to each other.

Definition 1.30. Let A and B be two finite graphs, and let M be a (possibly infinite) graph. The *relative dimension* of A over B in M, denoted $d_M(A/B)$, is given by the function

$$d_M(A/B) = d_M(AB) - d_M(B).$$

Remark 1.31. For any graphs A and B, $d(A/B) \geq 0$.

The relative dimension of A over B measures how much new information A adds to B, so if the vertices and edges of A are forced by B, then A does not add any new information to B and $d(A/B) = 0$, but if A has vertices and edges that are not forced by B, then A adds new information to B and $d(A/B) > 0$. Calculating the relative dimension, however, tends to be difficult, so we show three results that help us reduce the dimension to the predimension.

Lemma 1.32. $d(A/B) \leq \delta(A/\overline{B})$.

Proof. The graph $A\overline{B}$ contains AB, so $d(AB) \leq \delta(A\overline{B})$ and

$$d(A/B) = d(AB) - d(B) = d(AB) - \delta(\overline{B}) \leq \delta(A\overline{B}) - \delta(\overline{B}) = \delta(A/\overline{B}).$$

□

Proposition 1.33. $d(A/B) = \delta(\overline{AB}/\overline{B})$.

Proof. By Proposition 1.26, $\overline{B} \subseteq \overline{AB}$, so

$$d(A/B) = d(AB) - d(B) = \delta(\overline{AB}) - \delta(\overline{B}) = \delta(\overline{AB}/\overline{B}).$$

□

Corollary 1.34. *If $A \subseteq B$ are closed, then $d(B/A) = \delta(B/A)$.*

Proof. Since A, B are closed, then $B = \overline{B}$ and $A = \overline{A}$, so by Proposition 1.33,

$$\delta(B/A) = \delta(\overline{B}/\overline{A}) = \delta(\overline{BA}/\overline{A}) = d(B/A).$$

□

Closures preserve the relative dimension.

Corollary 1.35. $d(A/B) = d(\overline{A}/B) = d(A/\overline{B})$.

Proof. By Proposition 1.33 and Lemma 1.28,

$$\begin{aligned} d(A/B) &= \delta(\overline{AB}/\overline{B}) \\ &= \delta(\overline{\overline{A}B}/\overline{B}) = d(\overline{A}/B) \\ &= \delta(\overline{A\overline{B}}/\overline{\overline{B}}) = d(A/\overline{B}). \end{aligned}$$

□

To finish the chapter, we show three properties that allow us to reduce the relative dimension of a graph to the relative dimensions of its components.

Corollary 1.36. *[Dimension of unions]*

$$d(AB/C) = d(B/AC) + d(A/C).$$

Proof. By Proposition 1.26, $\overline{XY}\,\overline{X} = \overline{XY}$, so by Propositions 1.33 and 1.13,

$$\begin{aligned} d(AB/C) &= \delta(\overline{ABC}/\overline{C}) \\ &= \delta(\overline{ABC}\,\overline{AC}/\overline{C}) \\ &= \delta(\overline{ABC}/\overline{AC}\,\overline{C}) + \delta(\overline{AC}/\overline{C}) \\ &= \delta(\overline{ABC}/\overline{AC}) + \delta(\overline{AC}/\overline{C}) \\ &= d(B/AC) + d(A/C). \end{aligned}$$

□

Lemma 1.37. *If $B \subseteq C$ and $A \cap C = \emptyset$, then $d(A/C) \leq d(A/B)$.*

Proof. By Propositions 1.33 and 1.13,

$$d(A/C) = \delta(\overline{AC}/\overline{C}) = \delta(\overline{AB}\,\overline{AC}/\overline{C}) = \delta(\overline{AC}/\overline{AB}\,\overline{C}) + \delta(\overline{AB}/\overline{C}).$$

By Proposition 1.26, $\overline{AC}\,\overline{AB}\,\overline{C} = \overline{AC}$, and since $\delta(\overline{AC})$ is the infimum δ of a graph containing AC, then

$$\delta(\overline{AC}/\overline{AB}\,\overline{C}) = \delta(\overline{AC}\,\overline{AB}\,\overline{C}) - \delta(\overline{AB}\,\overline{C}) = \delta(\overline{AC}) - \delta(\overline{AB}\,\overline{C}) \leq 0,$$

so $d(A/C) \leq \delta(\overline{AB}/\overline{C})$. By Corollary 1.27, $\overline{AB} \cap \overline{C}$ is a closed graph containing B, so by Lemma 1.24, $\overline{AB} \cap \overline{C} = \overline{B}$, and by Corollary 1.11 and Proposition 1.33,

$$d(A/C) \leq \delta(\overline{AB}/\overline{C}) \leq \delta(\overline{AB}/\overline{B}) = d(A/B).$$

□

Corollary 1.38 (Subadditivity)**.** *If* $B \cap AC = \emptyset$*, then*

$$d(AB/C) \leq d(B/C) + d(A/C).$$

Proof. By Corollary 1.36 and Lemma 1.37,

$$d(AB/C) = d(B/AC) + d(A/C) \leq d(B/C) + d(A/C).$$

□

Chapter 2

The Collapsed Class

We have now built K and developed a basic understanding of the complexity of its graphs, so the next step is to use these graphs to build the infinite graph we are after. If we just "glue" all the graphs in K together into an infinite graph, however, the resulting graph would be so big that we would have no control over its structure. Like Borges' cartographers, we would have a map of the Empire that is the size of the Empire itself, so it would be useless. Instead, we need to find a subset of K that is large enough to suit our purposes, but small enough to have a manageable structure. Finding this subset is called "collapsing" K, and it is not easy. It is like taking K to the hospital. We need to identify K's disease, remove it, and then verify that K survived the surgery.

2.1 Simply Algebraic Extensions

In any case, let us start. The first step to collapsing K is understanding its disease, and the first step to understanding K's disease is to look at how we can extend its graphs without increasing their complexity. In particular, if we have $B \subseteq M \in K$, we want to know what the smallest extensions of B in M that preserve its complexity are. These extensions represent, in a way, the next steps we can take from B in M.

Definition 2.1. Let $B, C \in K$. The graph C is 0-*simply algebraic* over B if $\delta(C/B) = 0$ and $\delta(X/B) > 0$ for every $X \subsetneq C$.

Remark 2.2. If C is 0-simply algebraic over B, then $B \leq CB$.

The 0-simply algebraic extensions of B in M represent then the next steps B can take in M, so it makes sense for these next steps to be pairwise disjoint, either completely inside or completely outside strong graphs (since they do not increase complexity), and for their isomorphisms to extend over strong graphs. The next results show these properties.

Lemma 2.3. *Let $C \neq C'$ and $B \leq CC'$. If C and C' are 0-simply algebraic over B, then $(C - B) \cap (C' - B) = \emptyset$.*

Proof. Let $X = (C - B) \cap (C' - B)$. By Lemma 1.7,

$$\begin{aligned}\delta(CC'/B) &= \delta(CC'B) - \delta(B)\\ &= \delta(CC'BX) - \delta(C'BX) + \delta(C'BX) - \delta(BX) + \delta(BX) - \delta(B)\\ &= \delta(C/C'BX) + \delta(C'/BX) + \delta(X/B).\end{aligned}$$

By assumption, C is 0-simply algebraic over B, so $\delta(C/B) = 0$, and since $C \cap C'BX \subseteq BX$, then by Corollary 1.11 and Proposition 1.13,

$$\delta(C/C'BX) \leq \delta(C/BX) = \delta(C/B) - \delta(X/B) = -\delta(X/B),$$

and similarly $\delta(C'/BX) \leq -\delta(X/B)$, so

$$\delta(CC'/B) \leq -\delta(X/B) - \delta(X/B) + \delta(X/B) = -\delta(X/B).$$

By assumption, $B \leq CC'$, so $\delta(CC'/B) \geq 0$ and $\delta(X/B) \leq 0$. But $C \neq C'$, so without loss of generality $X \subsetneq C$, and since C is 0-simply algebraic over B, then $\delta(X/B) > 0$ if X is non-empty, so X must be empty. □

Proposition 2.4. *Let $F \subseteq B \leq D$ and $C \subseteq D$. If C is 0-simply algebraic over F, then $C \subseteq B$ or $C \subseteq D - B$.*

Proof. By Proposition 1.13,

$$0 = \delta(C/F) = \delta(C - B/(C \cap B)F) + \delta(C \cap B/F).$$

If towards contradiction $C \nsubseteq B$ and $C \nsubseteq D - B$, then $C \cap B$ is a non-empty proper subset of C, so $\delta(C \cap B/F) > 0$ and $\delta(C - B/(C \cap B)F) < 0$. But by Lemma 1.7 and Corollary 1.11,

$$\delta(C/B) = \delta(C - B/B) \leq \delta(C - B/(C \cap B)F) < 0$$

— a contradiction since $B \leq D$. □

Proposition 2.5. *Let $F \subseteq B \leq D, D'$, $C \subseteq D - B$, and $C' \subseteq D' - B$. If $C \cong_F C'$ and C is 0-simply algebraic over F, then C' is isomorphic to C over B.*

Proof. By assumption, $A \leq C$, so by Corollary 1.11,

$$0 \leq \delta(C/B) \leq \delta(C/F) = 0,$$

and by Proposition 1.9,

$$\begin{aligned}0 &= \delta(C/B)\\ &= \delta(C) - r(C,B)\\ &= \delta(C) - r(C,F) + r(C,F) - r(C,B)\\ &= \delta(C/F) + r(C,B-F)\\ &= r(C,B-F).\end{aligned}$$

Let $\varphi : CF \to C'F$ be the isomorphism fixing F. Since C is 0-simply algebraic over F, then $\varphi(C) = C'$ is 0-simply algebraic over F, so as above, $r(C', B-F) = 0$ and the map $\psi : CB \to C'B$ such that

$$\psi(x) = \begin{cases} x & x \in B \\ \varphi(x) & x \notin B, \end{cases}$$

is an isomorphism from C to C' fixing B. □

In most cases, however, we do not care about the next steps B can take in M, but rather about the new next steps, the next steps B can take in M that no $F \subsetneq B$ could take.

Definition 2.6. Let $B, C \in K$. The graph C is *minimally* 0*-simply algebraic* over B if C is 0-simply algebraic over B and C is not 0-simply algebraic over any $F \subsetneq B$.

Minimally 0-simply algebraic extensions capture the idea of the new next steps B can take in M, and they are of course correlated to 0-simply algebraic extensions. Intuitively, their correlation goes as follows. If B can take a next step C in M, then there must be some $F \subseteq B$ for which C was a new next step, and, in particular, such an F is unique.

Lemma 2.7. *If $F_1, F_2 \subseteq B$ and C is 0-simply algebraic over B, F_1, F_2, then C is* 0*-simply algebraic over $F_1 \cap F_2$.*

Proof. By Corollary 1.11 and Proposition 1.12,

$$\begin{aligned}0 &= \delta(C/F_1)\\ &\le \delta(C/F_1 \cap F_2)\\ &= \delta(C/F_1) + \delta(C/F_2) - \delta(C/F_1F_2)\\ &= -\delta(C/F_1F_2)\\ &\le -\delta(C/B)\\ &= 0,\end{aligned}$$

so $\delta(C/F_1 \cap F_2) = 0$, and since for any $X \subsetneq C$,

$$0 < \delta(X/F_1) \le \delta(X/F_1 \cap F_2),$$

then C is 0-simply algebraic over $F_1 \cap F_2$. □

Proposition 2.8. *Let $B, C \in K$. If C is 0-simply algebraic over B, then there is a unique $F \subseteq B$ such that C is minimally 0-simply algebraic over F.*

Proof. Let $\mathcal{F} = \{F \subseteq B \mid C \text{ is 0-simply algebraic over } F\}$. By assumption, $B \in \mathcal{F}$ and B is finite, so $\mathcal{F}$ is a non-empty, finite set. There must then be a subset minimal $F \in \mathcal{F}$. By construction, C is 0-simply algebraic over F and C is not 0-simply algebraic over any proper subset of F, so C is minimally 0-simply algebraic over F.

Suppose towards contradiction that there are $F_1 \neq F_2$ such that C is minimally 0-simply algebraic over F_1 and F_2, so $F_1 \cap F_2$ is a proper subset of either F_1 or F_2, and by Lemma 2.7, C is 0-simply algebraic over $F_1 \cap F_2$ — a contradiction. □

To finish the section, we introduce 1-simply algebraic extensions, since we can characterize them as a single vertex with a single edge, and they will help us simplify some proofs.

Definition 2.9. Let $B, C \in K$. The graph C is 1-*simply algebraic* over B if $\delta(C/B) = 1$ and $\delta(X/B) > 1$ for every $X \subsetneq C$ such that $X \subsetneq B$.

Proposition 2.10. *Let $B \subseteq C \in K$. If C is 1-simply algebraic over B, then $C = Bc$ and $r(c, B) = 1$.*

Proof. Let $X = C \setminus B$. By Lemma 1.7,

$$1 = \delta(C/B) = \delta(C - B/B) = \delta(X/B),$$

so X is a non-empty subset of C, and since $C \in K$, then $\delta(X) \geq 2$, so by Proposition 1.9 and Lemma 1.7,

$$r(X, B) = \delta(X) - \delta(X) + r(X, B) = \delta(X) - \delta(X/B) \geq 2 - \delta(C/B) = 1.$$

If towards contradiction $|X| > 1$, then $\delta(x/B) > 1$ for every $x \in X$, so

$$r(X, B) = \sum_{x \in X} r(x, B) = \sum_{x \in X} 2 - \delta(x/B) \leq 0$$

gives a contradiction. Thus, $|X| = 1$ and $r(X, B) = 1$. □

2.2 Collapsing K into K*

To construct the projective plane we are after, we need to be able to extend smaller graphs into bigger graphs without the complexity exploding. At first sight, minimally 0-simply algebraic extensions appear to be the perfect tool, but there is a problem. For any graph of K, there are too many minimally 0-simply algebraic extensions in K, too many possible new next steps. This abundance of minimally 0-simply algebraic extensions is K's disease, and to cure K of it, we need to remove some of them. But which ones should we remove? If we remove too many, then K may become a dull class without the properties we are after, and if we remove too few, then K may still suffer from the disease. We settle for a middle ground and just limit the number of repeated copies of each minimally 0-simply algebraic extension.

Definition 2.11. Let $C, F \in K$ where C is minimally 0-simply algebraic over F. The function f assigns a natural number $f(C/F) \geq 3$ to each pair (C, F), while the function μ is

$$\mu(C/F) = \begin{cases} 1 & \text{if } |C| = 1, |F| = 2, \text{and } r(C, F) = 2, \\ f(C/F) & \text{otherwise.} \end{cases}$$

For $B \in K$, the function χ_B assigns to each pair (C, F) the number of pairwise disjoint subgraphs $C' \subseteq B$ that are isomorphic to C over F.

The class $K^* \subseteq K$ is the collection of graphs $B \in K$ such that

$$\chi_B(C/F) \leq \mu(C/F).$$

We then limit the number of repeated copies of each minimally 0-simply algebraic extension to be a finite number (so that K is manageable) greater than 3 (so that K is not dull), and we add a special condition to preserve the "no-squares" rule.

This definition of K^*, however, is somewhat intricate, so to simplify future checks, we show the following results.

Lemma 2.12. *If $A \in K^*$, $r(b, A) \leq 2$, and there are no squares embedded in Ab, then $Ab \in K^*$.*

Proof. First we show that $B = Ab \in K$. By assumption, $A \in K^*$, so B is finite. Take $X \subseteq B$. If $X \subseteq A$, then $\delta(X) \geq 2$, so suppose that $X = X'b$ where $X' \subseteq A$. If $X' = \emptyset$, then $\delta(X) = 2 \geq 2$, and otherwise,

$$\delta(X) = \delta(X'b) = \delta(X') + \delta(b) - r(b, A') \geq \delta(X') \geq 2,$$

so $B \in K$.

Now we show that $B \in K^*$. Let $C, F \subseteq B$ be such that C is minimally 0-simply algebraic over F. If $\mu(C/F) = 1$, then $F = \{f_1, f_2\}$, $C = \{c\}$, and $r(c, f_1 f_2) = 2$. Any $c' \in B$ isomorphic to C over F would form a square in B, so $\chi_B(C/F) = 1$. Therefore, we may assume without loss of generality that $\mu(C/F) \geq 3$.

If $b \notin C, F$, then $C, F \subseteq A$, so $\chi_B(C/F) \leq \mu(C/F)$ and we are done. Suppose towards contradiction that $b \in C$. Since $\mu(C/F) \geq 3$, then $CF \neq bf_1 f_2$, so $X = C - b$ is a non-empty set such that $F \subsetneq X \subsetneq C$. Thus, $\delta(X/F) > 0$, and by Propositions 1.13 and 1.9,

$$\delta(C/F) = \delta(X/F) + \delta(b/XF) > \delta(b) - r(b, XF) \geq 2 - r(b, A) = 0$$

— a contradiction.

If $b \in F = F'b$, there are three cases.

- If towards contradiction $r(C, b) = 0$, then by Corollary 1.10, $\delta(C/F') = 0$ and $\delta(X/F') > 0$, so C is 0-simply algebraic over F' — a contradiction.
- If $r(C, b) = 1$, then b can only be related to a single $X \subseteq B \setminus C$ isomorphic to C over F, so $\chi_B(C/F) \leq 2$.
- If $r(C, b) = 2$, then b is unrelated to any $X \subseteq B \setminus C$, so $\chi_B(C/F) = 1$.

□

Corollary 2.13. *If $A \in K^*$ and $r(b, A) \leq 1$, then $Ab \in K^*$.*

Proof. Immediate from Lemma 2.12. □

After collapsing K into K^*, we would like to verify that K^* is neither too big nor too small. For now, however, we cannot really verify that K^* is not too big, so we will just verify that K^* is not too small. But verifying that K^* is not too small is not trivial, and it will take us many pages, so before diving into it, we perform two sanity checks.

Proposition 2.14. *If $A \cong B$ and $B \in K^*$, then $A \in K^*$.*

Proof. Let $f : A \to B$ be the isomorphism.

1. If there is a square S embedded in A, then $f(S)$ would be a square embedded in B, so A cannot contain any squares.
2. If there is a non-empty $X \subseteq A$ such that $\delta(X) \leq 1$, then $f(X)$ would be a non-empty subset of B such that $\delta(f(X)) \leq 1$, so $\delta(X) > 1$ for every non-empty $X \subseteq A$.
3. Suppose towards contradiction that there are $C, F \subseteq A$ such that C is minimally 0-simply algebraic over F and $\chi_A(C/F) > \mu(C/F)$. Denote $n = \mu(C/F) + 1$. There are $C_1, \ldots, C_n$ pairwise disjoint, isomorphic copies of C over F in A. The map f is an isomorphism, so $f(C_1), \ldots, f(C_n)$ are n-many pairwise disjoint, isomorphic copies of $f(C)$ over $f(F)$ in B, and we get a contradiction since then

$$\chi_B(f(C)/f(F)) \geq n > \mu(C/F) = \mu(f(F), f(C)).$$

□

Proposition 2.15. *If $A \subseteq B \in K^*$, then $A \in K^*$.*

Proof. Let $A \subseteq B \in K^*$.

1. If there is a square S embedded in A, then S would be a square embedded in B, so A cannot contain any squares.

2. If there is a non-empty $X \subseteq A$ such that $\delta(X) \leq 1$, then X would be a non-empty subset of B such that $\delta(X) \leq 1$, so $\delta(X) > 1$ for every non-empty $X \subseteq A$.

3. Suppose towards contradiction that there are $C, F \subseteq A$ such that C is minimally 0-simply algebraic over F and $\chi_A(C/F) > \mu(C/F)$. By assumption, $A \subseteq B$, so $\chi_A(C/F) \leq \chi_B(C/F)$ and, with it,

$$\chi_B(C/F) \geq \chi_A(C/F) > \mu(C/F)$$

— a contradiction since $B \in K^*$.

□

2.3 Amalgamation in K*

After the sanity checks, we need to show that K^* is not too small. More precisely, we need to verify that K^* satisfies the amalgamation property. For any two graphs $C, M \in K^*$ with some $B \leq C, M$ in common, we want to guarantee that there is a graph $N \in K^*$ that "preserves the structures" of C and M and "glues" them along B. But what does it mean to "preserve the structure" of C and M? And to "glue" C and M along B?

Let us focus on the first question. Usually, N preserves the structure of C if there is an embedding of C into N. But we have gone to great lengths defining and characterizing the complexity of graphs, so for N to preserve the structure of C, we also require N to preserve the complexity of C. We then say that an embedding *preserves the structure* of a graph if it preserves its complexity.

Definition 2.16. An embedding $f : B \to M$ is *strong* if $f(B) \leq M$.

We then call these structure preserving maps strong embeddings, and we can use them to define what does it mean to "glue" C and M along B, answering our second question.

Definition 2.17. Let $C, M \in K^*$ and $B \leq C, M$. An *amalgam* of C and M over B is a graph $M \leq N \in K^*$ for which there is a strong embedding $f : C \to N$ such that $f \upharpoonright B = \text{id}$.

A map N then *glues* C and M along B if N is an amalgam of C and M over B. For any two graphs, there are many different amalgams, but the simplest is attaching $C - B$ as a disjoint copy of M.

Definition 2.18. Let $C, M \in K^*$ and $B \leq C, M$. The *trivial amalgam* of C and M over B, denoted $C \otimes_B M$, is

$$C \otimes_B M = C' \cup M$$

where $C' = B \cup \{(0, c) \mid c \in C - B\}$, $C' \cong_B C$, and $r(C' - B, M - B) = 0$.

The trivial amalgam is, of course, an amalgam, and since it always exists, the difficulty of the next few pages lies in showing that the trivial amalgam of any two graphs $C, M \in K^*$ over a common graph $B \leq C, M$ is always in K^*. The proof is quite long, so to make it easier to follow, we split it into multiple cases using simply algebraic extensions.

We start by showing that the trivial amalgam is in K^* for 1-simply algebraic extensions.

Proposition 2.19. *Let $C, M \in K^*$ and $B \leq C, M$. If C is 1-simply algebraic over B, then $C \otimes_B M \in K^*$.*

Proof. By Proposition 2.10, $C = Bc$ where c is only related to some $b \in B$, so $C \otimes_B M \cong Mc$ where $r(M, c) = r(b, c) = 1$. The element c is related to a single element, so c cannot be in any square in Mc, and since $M \in K^*$, then there are no squares in Mc, so by Lemma 2.12, $Mc \in K^*$, and by Proposition 2.15, $C \otimes_B M \in K^*$. □

Next we show that the trivial amalgam is in K^* for 0-simply algebraic extensions. The proof is quite long, so to improve the readability, we split it into two lemmas with multiple claims, and a proposition.

Lemma 2.20. *Let $C \subseteq M \in K^*$ and $B \leq_C M$. If C is 0-simply algebraic over B, then either*

1. *$C \otimes_B M \in K^*$ or*
2. *$\chi_M(C - B/F) = \mu(C - B/F)$ for some $F \subseteq B$.*

Proof. Let $\hat{C} = C - B$ and $\hat{M} = M - B$. By Proposition 2.8, there is a unique $F \subseteq B$ such that $\hat{C}$ is minimally 0-simply algebraic over F, and since $B, C \subseteq M \in K^*$, then $\chi_M(\hat{C}/F) \leq \mu(\hat{C}/F)$. If $\chi_M(\hat{C}/F) = \mu(\hat{C}/F)$, we are in the second case, so we may assume that $\chi_M(\hat{C}/F) < \mu(\hat{C}/F)$, and we just need to show that $N = C \otimes_B M \in K^*$.

First we show that $N \in K$.

- There are no squares embedded in N.

  Suppose towards contradiction that there is a square S embedded in N. By assumption, $C, M \in K^*$, so S must contain vertices from both $\hat{C}$ and $\hat{M}$, and since $r(\hat{C}, \hat{M}) = 0$, then

$$S \cap \hat{C} = \{a\}, \quad S \cap \hat{M} = \{b\}, \quad S \cap B = \{c, d\}$$

where $r(a, cd) = r(b, cd) = 2$. By Corollary 1.11 and Proposition 1.9,

$$\delta(a/B) \leq \delta(a/S \cap B) = \delta(a) - r(a, S \cap B) = 0,$$

but C is 0-simply algebraic over B, so a cannot be a proper subset of $\hat{C}$ and $\hat{C} = a$. The element a is then minimally 0-simply algebraic over $S \cap B$, so $F = S \cap B$ by the uniqueness of F and $\mu(\hat{C}/F) = 1$ by the definition of μ. But by assumption $\chi_M(\hat{C}/F) < \mu(\hat{C}/F)$, so $\chi_M(\hat{C}/F) = 0$ — a contradiction since $\hat{C} \subseteq M$.

- For every $X \subseteq N$, $\delta(X) \geq 2$.

  Let $X \subseteq N$ and denote $X_M = X \cap M$ and $X_C = X - X_M$. The graph $\hat{C}$ is 0-simply algebraic over B and there are no edges between $\hat{C}$ and $\hat{M}$, so by Corollaries 1.10 and 1.11,

$$\delta(X_C/X_M) = \delta(X_C/X_M \cap B) \geq \delta(X_C/B) \geq 0,$$

  and with it,

$$\delta(X) = \delta(X_C X_M) = \delta(X_C/X_M) + \delta(X_M) \geq \delta(X_M) \geq 2.$$

Now we show that $N \in K^*$. Let $U, V \subseteq N$ be such that U is minimally 0-simply algebraic over V and $U \cap V = \emptyset$. We denote $\chi_N(U/V) = n$, so there are $U_1, \ldots, U_n \subseteq N$ pairwise disjoint, isomorphic graphs to U over V. For any set A, we denote

$$U_A = U_i \cap A \qquad \text{and} \qquad V_A = V \cap A.$$

The proof follows from Claims 3 and 5.

Claim 1. We can split the U_is into three categories.

$$\begin{cases} \delta(U_C/VU_B) < 0 & 1 \leq i \leq n_0 \\ U_M = U_i & n_0 < i \leq n_1 \\ U_M = \emptyset & n_1 < i \leq n \end{cases}$$

Proof. By Proposition 1.13 and Corollary 1.10,

$$\begin{aligned} 0 &= \delta(U/V) \\ &= \delta(U/VU_B) + \delta(U_B/V) \\ &= \delta(U_M/VU_B) + \delta(U_C/VU_M) + \delta(U_B/V) \\ &= \delta(U_M/VU_B) + \delta(U_C/VU_B) + \delta(U_B/V) \end{aligned}$$

so either

1. $\delta(U_C/VU_B) < 0$, or

2. $\delta(U_M/VU_B) + \delta(U_B/V) \le 0$.

For the second case, by Proposition 1.13,

$$\delta(U_M/V) = \delta(U_M/VU_B) + \delta(U_B/V) \le 0,$$

and since U_i is 0-simply algebraic over V, then $U_M = U_i$ or $U_M = \emptyset$. □

Claim 2. $n_0 \le \delta(V_C/B)$.

Proof. Fix $i \le n_0$. By Corollaries 1.10 and 1.11, Proposition 1.13, and Claim 1,

$$\delta(U_C/BV_C) = \delta(U_C/BV) \le \delta(U_C/U_BV) < 0.$$

Let $U^* = \bigcup_{i \le n_0} U_C$. By Corollary 1.14,

$$\delta(U^*/BV_C) \le \sum_{i \le n_0} \delta(U_C/BV_C) \le n_0 \cdot (-1) = -n_0,$$

and since $B \le C$ by Remark 2.2, then by Proposition 1.13,

$$\delta(V_C/B) = \delta(U^*V_C/B) - \delta(U^*/BV_C) \ge \delta(U^*V_C/B) + n_0 \ge n_0.$$

□

Claim 3. If $V \subseteq M$, then $\chi_N(U/V) \le \mu(U/V)$.

Proof. Since $V \subseteq M$, then $V_C = V_B \subseteq B$, so by Claim 2,

$$0 \le n_0 \le \delta(V_C/B) = \delta(V_B/B) = 0,$$

and $n_0 = 0$. There are thus two cases.

- If $n_1 = n$, then every $U_i = U_M \subseteq M$, and since $M \in K^*$, we are done.
- If $n_1 < n$, then $U_n \subseteq \hat{C}$. By construction, $r(\hat{C}, \hat{M}) = 0$ and U_n is minimally 0-simply algebraic over V, so by Corollaries 1.11 and 1.10,

  $$\delta(U_n/B) \le \delta(U_n/V_B) = \delta(U_n/V_M) = \delta(U_n/V) = 0.$$

  But by assumption $\hat{C}$ is 0-simply algebraic over B, so U_n cannot be a proper subset of $\hat{C}$, so $U_n = \hat{C}$, and since the different U_is are disjoint, then $n_1 = n - 1$.

  An argument similar to the above shows that U_n is 0-simply algebraic over V_B, so $V = V_B$ by the minimality of V. But F is the unique subgraph of B such that $\hat{C}$ is minimally 0-simply algebraic over F, so $F = V$, and since by assumption $\chi_M(\hat{C}/F) < \mu(\hat{C}/F)$, then

$$\begin{aligned}\chi_N(U/V) &= \chi_N(U_n/V) \\ &= \chi_N(\hat{C}/F) \\ &= \chi_M(\hat{C}/F) + 1 \\ &\leq \mu(\hat{C}/F) \\ &= \mu(U_n/V) \\ &= \mu(U/V).\end{aligned}$$

□

Claim 4. If $V \nsubseteq M$, then $n_1 \leq \delta(V_C/V_B)$.

Proof. Let $V_{\hat{C}} = V \cap \hat{C}$. Fix $n_0 < i \leq n_1$. By assumption, $r(\hat{C}, \hat{M}) = 0$, so by Claim 1,

$$0 \leq r(U_i, V_{\hat{C}}) = r(U_M, V_{\hat{C}}) = r(U_B, V_{\hat{C}}).$$

If towards contradiction $r(U_i, V_{\hat{C}}) = 0$, then by Corollary 1.10, since U_i is 0-simply algebraic over V, U_i is 0-simply algebraic over V_M. But $V \nsubseteq M$, so U_i is 0-simply algebraic over a proper subset of V — a contradiction since V is minimal. Thus, $r(U_i, V_{\hat{C}}) \geq 1$.

All the U_is are disjoint from V and from each other, so

$$n_1 - n_0 \leq \sum_{n_0 < i \leq n_1} r(V_{\hat{C}}, U_B) \leq r(V_{\hat{C}}, B - V_B) = r(V_{\hat{C}}, B) - r(V_{\hat{C}}, V_B),$$

and by Lemma 1.7 and Proposition 1.9,

$$\begin{aligned}n_1 - n_0 &\leq r(V_{\hat{C}}, B) - r(V_{\hat{C}}, V_B) \\ &= r(V_{\hat{C}}, B) - \delta(V_{\hat{C}}) + \delta(V_{\hat{C}}) - r(V_{\hat{C}}, V_B) \\ &= \delta(V_{\hat{C}}/V_B) - \delta(V_{\hat{C}}/B) \\ &= \delta(V_C/V_B) - \delta(V_C/B),\end{aligned}$$

so by Claim 2,

$$n_1 = (n_1 - n_0) + n_0 \leq \delta(V_C/V_B) - \delta(V_C/B) + \delta(V_C/B) = \delta(V_C/V_B).$$

□

Claim 5. If $V \nsubseteq M$, then $\chi_N(U/V) \leq \mu(U/V)$.

Proof. There are two cases.

- Suppose that $n_1 < n$.

  By Claim 1, $U_n \subseteq \hat{C}$, and since $r(\hat{C}, \hat{M}) = 0$ and U_n is 0-simply algebraic over V, then by Corollary 1.10, the graph U_n is 0-simply algebraic over V_C, so $V = V_C$ by the minimality of V.

  If we can also show that $U_i \subseteq C$, then since $C \in K^*$, we would be done, so suppose towards contradiction that there is $U_i \nsubseteq C$. Then U_C is a proper subset of U_i, and since U_i is 0-simply algebraic over V, then $\delta(U_C/V) > 0$, so by Propositions 1.13 and Corollary 1.10,

  $$0 = \delta(U_i/V) = \delta(U_C/V) + \delta(U_M/VU_C) > \delta(U_M/VU_B).$$

  But U_i is isomorphic to C_n and $B \leq_C M$, so by Corollary 1.10

  $$0 \leq \delta(U_{\hat{M}}/V_B U_B) = \delta(U_{\hat{M}}/VU_B)$$

  — a contradiction.

- Suppose that $n_1 = n$.

  By definition, $C \leq N$, so by Lemma 1.7 and Corollary 1.11,

  $$0 \leq \delta(V/C) = \delta(V_M/C) \leq \delta(V_M/V_C).$$

  The graph V_B is a subset of $B \subseteq C \in K^*$, so $\delta(V_B) \geq 0$, and by Claim 4,

  $$\begin{aligned} n &= n_1 \\ &\leq \delta(V_C/V_B) \\ &\leq \delta(V_C/V_B) + \delta(V_M/V_C) + \delta(V_B) \\ &= \delta(V_M V_C) \\ &= \delta(V) \\ &\leq \mu(U/V). \end{aligned}$$

□

□

Lemma 2.21. *Let $C, M \in K^*$ and $B \leq C, M$. If C is 0-simply algebraic over B and there is $F \subseteq B$ such that $\chi_M(C - B/F) = \mu(C - B/F)$, then there is $C' \leq M$ isomorphic to C over B.*

Proof. Consider the set

$$\mathcal{D} = \{D \subseteq M \mid D \cong_F C - B\}.$$

By Proposition 2.4, every $D \in \mathcal{D}$ is either a subset of $M \setminus B$ or of B. If towards contradiction every $D \in \mathcal{D}$ is a subset of B, then

$$\chi_C(C-B/F) = \chi_B(C-B/F)+1 = \chi_M(C-B/F)+1 = \mu(C-B/F)+1,$$

contradicting $\chi_C(C-B/F) \leq \mu(C-B/F)$. There is then some $D \in \mathcal{D}$ such that $D \subseteq M \setminus B$. By Proposition 2.5, $DB \cong_B C$, and since $\delta(D/B) = 0$, then by Lemma 1.18, $DB \leq M$. □

Proposition 2.22. *Let $C, M \in K^*$ and $B \leq C, M$. If C is 0-simply algebraic over B, then either $C \otimes_B M \in K^*$ or M is an amalgam of C and M over B.*

Proof. By Lemma 1.21, $B \leq_C M$, so by Lemma 2.20, either $C \otimes_B M \in K^*$ or there is $F \subseteq B$ such that $C - B$ is minimally 0-simply algebraic over F and $\chi_M(C-B/F) = \mu(C-B/F)$. If we are in the first case, we are done, and if we are in the second case, by Lemma 2.21, there is $C' \leq M$ isomorphic to C over B, so $M \in K^*$ is an amalgam of C and M over B. □

We finally put these results together to prove the general case.

Theorem 2.23 (Amalganation Property)**.** *If $C, M \in K^*$ and $B \leq C, M$, then there is an amalgam $N \in K^*$ of C and M over B.*

Proof. Let $C, M \in K^*$ and $B \leq C, M$. If $C = M$, then C is the amalgam, and if $C = B$ (or $M = B$), then M (or C) is the amalgam, so we may assume without loss of generality that $B \subsetneq C, M$ and $C \neq M$. In fact, since C and M are isomorphic to

$$C' = B \cup \{(0,c) \mid c \in C - B\} \quad \text{and} \quad M' = B \cup \{(1,c) \mid c \in M - B\},$$

any amalgam of C' and M' over B is also an amalgam of C and M over B, so we may assume without loss of generality that $C \cap M = B$.

We show by induction on $|CM - B|$ that there is an amalgam $N \in K^*$ of C and M over B such that $|N| \leq |CM|$.

- If $|CM - B| = 0$, then $M = C = B$, so we are done by the above.
- If $|CM - B| = n+1$, there are two cases.

  1. There is $X \subsetneq C - B$ such that $\delta(X/B) = 0$.
     By assumption, $C \cap M = B$, so

     $$|XM - B| < |CM - B| = n+1,$$

     and by the induction hypothesis there is an amalgam $N' \in K^*$ of XB and M over B such that $|N'| \leq |MX|$. By assumption, $\delta(X/B) = 0$, so by Lemma 1.18, $XB \leq C$, and since

     $$\left|CN' - XB\right| \leq |CM - XB| < |CM - B| = n+1,$$

then by the induction hypothesis there is an amalgam $N \in K^*$ of C and N' over XB such that $|N| \leq |CN'|$.

By construction, $M \leq N' \leq N$, so by Lemma 1.17, $M \leq N$, and since $C \leq N$, then $N \in K^*$ is an amalgam of C and M over B such that

$$|N| \leq \left|CN'\right| \leq |CMX| = |CM|\,.$$

2. $\delta(X/B) > 0$ for every $X \subsetneq C - B$.

   There are three subcases.

   (a) If $\delta(C/B) = 0$, then C is 0-simply algebraic over B, so we are done by Proposition 2.22.

   (b) If $\delta(C/B) = 1$, there are two subsubcases. If $\delta(X/B) > 1$ for every $X \subsetneq C - B$, then C is 1-simply algebraic over B, so we are done by Proposition 2.19. If $\delta(X/B) = 1$ for some $X \subsetneq C - B$, then by Proposition 1.13 and the assumption that $\delta(Y/B) \geq 1$ for every other $Y \subsetneq C - B$,

   $$\delta(Y/CB) = \delta(YC/B) - \delta(C/B) \geq \delta(YC/B) - 1 \geq 0,$$

   so $XB \leq C$ and the proof follows as in Case 1.

   (c) Suppose that $\delta(C/B) \geq 2$. By assumption $C \neq B$, so $C - B$ is non-empty and there is $c \in C - B$. If we can show that there is an amalgam $N \in K^*$ of Bc and M over B such that $|N| \leq |Mc|$, then the result follows as in Case 1.

   There are three subsubcases.

   i. If $\delta(c/B) = 0$, then c is 0-simply algebraic over B, so we are done by Proposition 2.22.

   ii. If $\delta(c/B) = 1$, then c is 1-simply algebraic over B, so we are done by Proposition 2.19.

   iii. If $\delta(c/B) \geq 2$, then by Proposition 1.9,

   $$0 \leq r(c, B) = \delta(c) - \delta(c/B) \leq 0,$$

   so $r(c, B) = 0$. Let $N = Mc$ where $r(M, c) = 0$. By Lemma 2.12, $N \in K^*$, and since $Bc \leq N$ and $M \leq N$, then N is an amalgam of Bc and M over B.

□

The following corollary is immediate in our set-up.

Corollary 2.24 (Joint Embedding Property). *If $B, C \in K^*$, then there are strong embeddings from B and C into some $N \in K^*$.*

Proof. Let $B, C \in K^*$. By the definition of K^*, $\delta(X) \geq 2$ for any $X \subseteq B, C$, so $\emptyset \leq B, C$. By Theorem 2.23, there is an amalgam $N \in K^*$ of B and C over $\emptyset$, so there are strong embeddings of B and C into N. □

Chapter 3

The Universal-Homogeneous Graph

We now have all the ingredients we need to build the projective plane we are after, and as we anticipated, it will be an infinite graph.

3.1 Construction of M

The projective plane we are after will, in particular, be the infinite graph that is K^*-universal, meaning that it contains everything in K^* and nothing else, and K^*-homogeneous, meaning that it looks the same everywhere. We will then say that this graph is K^*-universal-homogeneous, but note that some mathematicians would refer to it as the Fraïssé limit of the class K^*.

Definition 3.1. A countable graph M is *K^*-universal-homogeneous* if

1. Every $A \leq M$ is in K^*.
2. For every finite $A \subseteq M$, there is $B \in K^*$ such that $A \subseteq B \leq M$.
3. For every $A \in K^*$, there is a strong embedding $f : A \to M$.
4. For all $A, B \in K^*$ such that $A \leq M$ and $A \leq B$, there is a strong embedding $f : B \to M$ such that $f \restriction A = \mathrm{id}$.

Our first task is to show that there exists such a graph, and we construct it by gluing together all the graphs in K^*.

Proposition 3.2. *There is a K^*-universal-homogeneous graph.*

Proof. Let $\{(A_n, B_n) \in (K^*)^2 \mid n \in \mathbb{N}\}$ be a set such that

1. For every $n \in \mathbb{N}$, $\mathrm{dom}(A_n) \subseteq \mathbb{N}$, $\mathrm{dom}(B_n) \subseteq \mathbb{N}$, and $A_n \leq B_n$.
2. For every $A \in K^*$, there is $n \in \mathbb{N}$ such that $A \cong A_n$.

3. For every $n \in \mathbb{N}$ and $A, B \in K^*$ such that $A \leq B$ and $\mathrm{dom}(A) \subseteq \mathbb{N}$, there is $k \in \mathbb{N}$ such that $k > n$, $A_k = A$, and there is an isomorphism $f : B \to B_k$ such that $f \restriction A = \mathrm{id}$.

We construct by recursion on $\mathbb{N}$ a set $\{M_n \in K^* \mid \mathrm{dom}(M_n) \subseteq \mathbb{N}, n \in \mathbb{N}\}$.

- We define $M_0 = A_0$.
- Suppose that we have constructed M_n.
  - Suppose that $A_n \leq M_n$. By construction, $A_n \in K^*$, and, by the induction hypothesis, $M_n \in K^*$. By Theorem 2.23, there is $N \in K^*$ and strong embeddings $f : B_n \to N$ and $g : M_n \to N$ such that $f \restriction A_n = g \restriction A_n$. Let $M_{n+1} \in K^*$ be such that $M_{n+1} \cong N$, $M_n \subseteq M_{n+1}$, and $\mathrm{dom}(M_{n+1}) \subseteq \mathbb{N}$. Denote by $h : N \to M_{n+1}$ the isomorphism such that $h \circ g = \mathrm{id}$. The map $h \circ f$ is a strong embedding of B_n into M_{n+1} such that $(h \circ f) \restriction A_n = \mathrm{id}$, and $M_n \leq M_{n+1}$ since
  $$M_n = (h \circ g)(M_n) \leq h(N) = M_{n+1}.$$
  - Suppose that $A_n \not\leq M_n$. By construction, $A_n \in K^*$, and, by the induction hypothesis, $M_n \in K^*$. By Corollary 2.24, there is $N \in K^*$ and strong embeddings $f : A_n \to N$ and $g : M_n \to N$. As above, there is $M_{n+1} \in K^*$ such that $M_{n+1} \cong N$, $M_n \leq M_{n+1}$, $\mathrm{dom}(M_{n+1}) \subseteq \mathbb{N}$, and there is a strong embedding $h : A_n \to M_{n+1}$.

We define $M = \bigcup_{n \in \mathbb{N}} M_n$. The graph M is the countable union of finite graphs, so it is countable, and it is K^*-universal-homogeneous since:

1. Take $A \leq M$. By construction, there is $n \in \mathbb{N}$ such that $A \subseteq M_n \in K^*$, so, by Proposition 2.15, $A \in K^*$.

2. Take $A \subseteq M$ finite. By construction, there is $n \in \mathbb{N}$ such that $A \subseteq M_n \in K^*$. If towards contradiction $M_n \not\leq M$, then there is $B \subseteq M_k$ for $k > n$ witnessing it, so $M_n \not\leq M_k$ — contradicting Lemma 1.17. Thus, $M_n \leq M$.

3. Take $A \in K^*$. By definition, there is $n \in \mathbb{N}$ such that $A \cong A_n$. Define $g : A \to A_n$ to be the isomorphism. If $A_n \leq M_n$, then, by Lemma 1.17, $A_n \leq M$, so g is a strong embedding of A into M. If $A_n \not\leq M_n$, then there is a strong embedding $h : A_n \to M_{n+1}$, so $h \circ g$ is a strong embedding of A into M.

4. Take $A, B \in K^*$ such that $A \leq M$ and $A \leq B$. By construction, there is $n \in \mathbb{N}$ such that $A \subseteq M_n$ and, since $\mathrm{dom}(M_n) \subseteq \mathbb{N}$, then

$\mathrm{dom}(A) \subseteq \mathbb{N}$. By definition, there is then $k \in \mathbb{N}$ such that $k > n$, $A_k = A$, and there is an isomorphism $f : B \to B_k$ such that $f \restriction A = \mathrm{id}$. By assumption, $A \leq M$ and $A \subseteq M_n$, so $A \leq M_n$ and, since $k > n$, then $A_k \leq M_k$. There is then a strong embedding $g : B_k \to M_{k+1}$ such that $g \restriction A_k = \mathrm{id}$, so $g \circ f$ is a strong embedding of B into M such that

$$(g \circ f)(A) = g(A) = g(A_k) = A_k = A.$$

□

Next, we need to show that K^*-universal-homogeneous graphs are homogeneous, which is not obvious from the definition. The point is that K^*-universal-homogeneous graphs have such a rigid structure that, once we know where a single strong subgraph is, we can reconstruct the whole graph, like a puzzle.

Proposition 3.3. *If M, N are K^*-universal-homogeneous graphs, $A \leq M$, $B \leq N$, and $f : A \to B$ is an isomorphism, then there is an isomorphism $g : M \to N$ such that $f \subseteq g$.*

Proof. Let M, N be K^*-universal-homogeneous graphs. Take $A \leq M$ and $B \leq N$ such that there is an isomorphism $f : A \to B$. By the definition of K^*-universal-homogeneous, the graphs M, N are countable, so we can enumerate them as

$$M = \{a_n \mid n \in \mathbb{N}\} \quad \text{and} \quad N = \{b_n \mid n \in \mathbb{N}\}.$$

If we can construct a set $\{(A_n, B_n, f_n) \mid n \in \mathbb{N}\}$ such that, for every $n \in \mathbb{N}$,

1. $A_n \leq M$, $B_n \leq N$, and $f_n : A_n \to B_n$ is an isomorphism.
2. If $i < j$, then $A_i \subseteq A_j$, $B_i \subseteq B_j$, and $f_i \subseteq f_j$.
3. If $n = 2k + 1$, then $a_k \in A_n$, and if $n = 2k + 2$, then $b_k \in B_n$.
4. $A_0 = A$, $B_0 = B$, and $f_0 = f$.

The map $g = \bigcup_{n\in\mathbb{N}} f_n$ would then be an isomorphism from $\bigcup_{n\in\mathbb{N}} A_n = M$ to $\bigcup_{n\in\mathbb{N}} B_n = N$ such that $f \subseteq g$, so we would be done.

We construct the set by induction on $n \in \mathbb{N}$.

- If $n = 0$, we define $A_0 = A$, $B_0 = B$, and $f_0 = f$.
- Suppose that $n+1 = 2k+1$ and that we have constructed (A_n, B_n, f_n). By the induction hypothesis, $A_n \leq M$, so $A_n \subseteq M$ and, with it, $A_n \cup \{a_k\} \subseteq M$. By the definition of K^*-universal-homogeneous, there is then

$$A_{n+1} \in K^* \quad \text{such that} \quad A_n \cup \{a_k\} \subseteq A_{n+1} \leq M.$$

By copying the new vertices and relations added to A_{n+1}, we can find an extension B' of B_n isomorphic to A_{n+1}, so there is an isomorphism $g : A_{n+1} \to B'$ such that $f_n \subseteq g$. We have that $A_n, A_{n+1} \leq M$ and $A_n \subseteq A_{n+1}$, so $A_n \leq A_{n+1}$ and, with it, $B_n \leq B'$. By construction, $B' \cong A_{n+1} \in K^*$, so, by Proposition 2.15, $B' \in K^*$. By the induction hypothesis, $B_n \leq N$, so, by the definition of K^*-universal-homogeneous, there is a strong embedding $h : B' \to N$ such that $h \restriction B_n = \mathrm{id}$, and we define

$$f_{n+1} = h \circ g \quad \text{and} \quad B_{n+1} = f_{n+1}(A_{n+1}).$$

- Suppose that $n+1 = 2k+2$ and that we have constructed (A_n, B_n, f_n). The proof is similar to the above, but reversed. We find $B_{n+1} \in K^*$ such that $B_n \cup \{b_k\} \subseteq B_{n+1} \leq N$, we construct an extension A' of A_n isomorphic to B_{n+1}, we find an isomorphism $g : B_{n+1} \to A'$ such that $f_n \subseteq g$ and a strong embedding $h : A' \to M$ such that $h \restriction A_n = \mathrm{id}$, and we define

$$f_{n+1} = h \circ g \quad \text{and} \quad A_{n+1} = f_{n+1}(B_{n+1}).$$

To verify that the set $\{(A_n, B_n, f_n) \mid n \in \mathbb{N}\}$ is as we wanted, notice that properties 3 and 4 follow immediately by construction, so we just need to verify properties 1 and 2.

1. If $n = 0$, then by assumption $A_0 = A \leq M$, $B_0 = B \leq N$, and $f_0 = f$ is an isomorphism. If $n > 0$, then by construction $A_n \leq M$, $B_n \leq N$, and f_n is the composition of an isomorphism and an embedding, so it is an isomorphism onto its image.

2. If $i < j$, the different cases are similar, so we can suppose without loss of generality that $i = 2k$ and $j = 2k + 1$. By construction, $A_i \cup \{a_k\} \subseteq A_j$, so $A_i \subseteq A_j$. By construction, $f_j = h \circ g$, where $f_i \subseteq g$, $g(A_i) = B_i$, and $h \restriction B_i = \mathrm{id}$, so $f_i \subseteq f_j$. Finally,

$$B_i = f_i(A_i) = f_j(A_i) \subseteq f_j(A_j) = B_j.$$

□

Corollary 3.4 (Homogeneity)**.** *If M is a K^*-universal-homogeneous graph, $A, B \leq M$ and $f : A \to B$ is an isomorphism, then there is an automorphism g of M such that $f \subseteq g$.*

Proof. This result is the special case of Proposition 3.3 where $N = M$. □

Finally, we can show that K^*-universal-homogeneous graphs are unique up to isomorphism.

Corollary 3.5. *Any two K^*-universal-homogeneous graphs are isomorphic.*

Proof. Let M, N be two K^*-universal-homogeneous graphs. Take $A \in K^*$. By definition, there are strong embeddings $f : A \to M$ and $g : A \to N$, so the map $g \circ (f^{-1} \upharpoonright f(A))$ is an isomorphism from $f(A) \leq M$ to $g(A) \leq N$, and by Proposition 3.3, there is an isomorphism from M to N. □

Now, we can fix a K^*-universal-homogeneous graph and denote it M. We will think of M as a structure in the language with a single binary relation R such that, for all $a, b \in M$,

$$aRb \iff r(a, b) = 1.$$

3.2 M as a Projective Plane

The graph M is the infinite graph we were after, so we now need to show that it has the properties we wanted. We then need to show that M is a projective plane, that we can define the elements of M with formulas with parameters over a strongly minimal set, and that M does not admit a coordinate system over a division ring.

Let us start by looking at the geometry of M.

Definition 3.6. We call the elements of M *points* and the subsets

$$\ell_b := \{a \in M \mid aRb\},$$

where $b \in M$, *lines.* We say that a point $a \in M$ *lies on* a line ℓ_b if $a \in \ell_b$.

We will shortly start working with different models. Most of the time, it will be clear from the context to what model a point or a line belong to, but when it is not, we will write a^M or ℓ_b^M to clarify.

Before trying to understand the geometry of M, we need to show a few properties about how complexity behaves in the lines of M.

Lemma 3.7. *If $Xb \subseteq M$ is finite and $a \in \ell_b$, then*

$$0 \leq d(a/Xb) \leq \delta(a/\overline{Xb}) \leq 1.$$

Proof. By Lemma 1.32,

$$0 \leq d(a/Xb) \leq \delta(a/\overline{Xb}).$$

If $a \in \overline{Xb}$, then $\delta(a/\overline{Xb}) = 0$, and if $a \notin \overline{Xb}$, then by Proposition 1.9,

$$\delta(a/\overline{Xb}) = \delta(a) - r(a, \overline{Xb}) \leq 2 - r(a, b) = 1.$$

□

Lemma 3.8. *If $Xb \subseteq M$ is finite and $a \in \ell_b$ is such that $d(a/Xb) = 1$, then $\overline{Xb}a \leq M$.*

Proof. By Corollary 1.35 and Lemma 3.7,

$$\begin{aligned} d(\overline{Xb}a) &= d(a/\overline{Xb}) + d(\overline{Xb}) \\ &= d(a/Xb) + \delta(\overline{Xb}) \\ &= \delta(a/\overline{Xb}) + \delta(\overline{Xb}) \\ &= \delta(\overline{Xb}a), \end{aligned}$$

so by Proposition 1.23, $\overline{Xb}a \leq M$. □

Lemma 3.9. *If $Xa \subseteq \ell_b$ is finite and $|X| \geq 3$, then $b \in \overline{X}$.*

Proof. By Proposition 1.9,

$$\delta(b/X) = \delta(b) - r(b, X) = \delta(b) - |X| \leq 2 - 3 < 0,$$

so by Lemma 1.25, $b \in \overline{X}$. □

Proposition 3.10. *If $Xb \subseteq M$ is finite and there are $a, c \in \ell_b$ such that $d(a/Xb) = d(c/Xb) = 1$, then there is an automorphism of M fixing $\overline{Xb}$ and sending a to c.*

Proof. By Lemma 3.7, $\delta(a/\overline{Xb}) = \delta(c/\overline{Xb}) = 1$, so a and c are only related to b in $\overline{Xb}$. There is then an isomorphism from $\overline{Xb}a$ to $\overline{Xb}c$ fixing $\overline{Xb}$, and since $\overline{Xb}a \leq M$ and $\overline{Xb}c \leq M$ by Lemma 3.8, then by Corollary 3.4, there is an automorphism of M sending a to c and fixing $\overline{Xb}$. □

Now that we have a better understanding of how complexity behaves in the lines of M, we can start to investigate the geometry of M as a whole. In particular, we can show one of the main results we are after, that M is a projective plane.

Definition 3.11. A *projective plane* is a collection of points and lines together with an incidence relation such that:

- any two distinct lines intersect at a unique point,
- any two distinct points lie on a unique line,
- there are four points such that no three points lie on the same line.

The incidence relation between the points and lines of M is the relation we previously defined as *lies on*. Using two lemmas, we can then show that M is a projective plane.

Lemma 3.12. *If $a, b \in M$ are distinct, then there is a unique $c \in M$ such that aRc and bRc.*

Proof. Let $A = ab \subseteq M$ with $a \neq b$. If there is $c \in \overline{A}$ such that aRc and bRc, then we are done, so suppose that there is no such c. Let $B = \overline{A}c$ where aRc, bRc, and $r(\overline{A}, c) = 2$. There are no squares in B since $\overline{A} \in K^*$ and c is only related to a and b, which are not both related to any point in $\overline{A}$, so by Lemma 2.12, $B \in K^*$. By construction, $\overline{A} \leq B$, and since $\overline{A} \leq M$, there is a strong embedding $f : B \to M$ such that $f \restriction \overline{A} = \mathrm{id}$, so $f(c) = c' \in M$ is such that aRc' and bRc'.

Any such element is unique since, if there were two such elements $c, c' \in M$, then the graph $cac'b$ would form a square in M. □

Lemma 3.13. *There is an infinite graph $D \subseteq M$ such that*

$$d(X) = \delta(X) = 2\,|X| \qquad \forall X \subseteq D \text{ finite.}$$

Proof. We construct by recursion on $n \in \mathbb{N}$ a collection $\{D_n \leq M \mid n \in \mathbb{N}\}$ such that, for every $n \in \mathbb{N}$, $|D_n| = n + 1$ and $e(D_n) = 0$.

- Let $D_0 = a$ for any $a \in M$.
- Suppose that we have constructed D_n. Let $A = D_n a$ be such that $r(a, D_n) = 0$. By construction, $A \in K^*$, $D_n \leq M$, and $D_n \leq A$, so there is a strong embedding $f : A \to M$ such that $f \restriction D_n = \mathrm{id}$. Let $D_{n+1} = f(A) = D_n b$.

The graph $D = \bigcup_{n \in \mathbb{N}} D_n$ is an infinite subset of M such that, for every finite $X \subseteq D$, there is $n \in \mathbb{N}$ such that $X \subseteq D_n \leq M$ and $e(D_n) = 0$, so $X \leq D_n$, and by Lemma 1.17 and Proposition 1.23, $d(X) = \delta(X) = 2\,|X|$. □

Theorem 3.14. *M is a projective plane.*

Proof. We show that M satisfies the conditions defining a projective plane.

- Any two distinct lines intersect at a unique point.

  Take two distinct lines $\ell_a, \ell_b \subseteq M$. By Lemma 3.12, there is a unique d related to both a and b, so $d \in \ell_a \cap \ell_b$ and, by the uniqueness of d, there is no such other point.

- Any two distinct points lie on a unique line.

  Take two distinct points $a, b \in M$. By Lemma 3.12, there is a unique d related to both a and b, so $a, b \in \ell_d$ and, by the uniqueness of d, there is no such other line.

- There are four points with no three points lying on a line.

  By Lemma 3.13, there is an infinite graph $D \subseteq M$ such that $d(X) = \delta(X) = 2\,|X|$ for any finite $X \subseteq D$. Take $X \subseteq D$ with $|X| = 4$. If towards contradiction there is $Y \subseteq X$ such that $|Y| \geq 3$ and $Y \subseteq \ell_b$ for some $b \in M$, then by Lemma 3.9, $b \in \overline{X}$ — a contradiction since X is closed.

□

3.3 Elementary Extensions of M

The model M is then a projective plane, so its geometric structure is well understood. But is its geometric structure a property of M, or of its theory? Is M an exceptional model, or is it part of a collection of models with the same properties? To answer this question, we need to look at the elementary extensions of M, the models that satisfy the same first-order theory as M.

Definition 3.15. A structure N is an *elementary extension* of M, denoted $M \preccurlyeq N$, if $M \subseteq N$ and

$$M \models \varphi(a_1, \dots, a_n) \iff N \models \varphi(a_1, \dots, a_n)$$

for any formula $\varphi(x_1, \dots, x_n)$ and any collection of points $a_1, \dots, a_n \in M$.

We denote by N an arbitrary, countable elementary extension of M.

Remark 3.16. If $N \preccurlyeq P$ and $P \preccurlyeq Q$, then $N \preccurlyeq Q$.

Lemma 3.17. *If $\{N_i\}_{i\in\mathbb{N}}$ is a chain of structures such that $N_i \preccurlyeq N_{i+1}$ for every $i \in \mathbb{N}$, then for every $i \in \mathbb{N}$,*

$$N_i \preccurlyeq \bigcup_{j\in\mathbb{N}} N_j.$$

Proof. Let $N = \bigcup_{j\in\mathbb{N}} N_j$. To show that for every $i \in \mathbb{N}$, $N_i \preccurlyeq N$, we show by induction on formulas $\varphi(x)$ that, for all $i \in \mathbb{N}$ and $a \in N_i$,

$$N_i \models \varphi(a) \iff N \models \varphi(a).$$

- If $\varphi(x)$ is an atomic formula, then φ is either an equality or a relation. If φ is an equality, the claim is clear, and if φ is a relation, then since $N_i \subseteq N$, the relations between elements of N_i are the same in N_i and in N, so the claim holds.
- If $\varphi(x) = \neg\psi(x)$, then by the induction hypothesis,

$$N_i \models \neg\psi(a) \iff N_i \not\models \psi(a) \iff N \not\models \psi(a) \iff N \models \neg\psi(a).$$

- If $\varphi(x) = \psi(x) \wedge \theta(x)$, then by the induction hypothesis,

$$\begin{aligned} N_i \models \psi(a) \wedge \theta(a) &\iff N_i \models \psi(a) \text{ and } N_i \models \theta(a) \\ &\iff N \models \psi(a) \text{ and } N \models \theta(a) \\ &\iff N \models \psi(a) \wedge \theta(a). \end{aligned}$$

- Suppose that $\varphi(x) = \exists y\, \psi(y, x)$.
  - If $N_i \models \varphi(a)$, then there is $y \in N_i$ such that $N_i \models \psi(y, a)$, and since $N_i \subseteq N$, then $y \in N$, so by the induction hypothesis, $N \models \psi(y, a)$ and $N \models \varphi(a)$.

– If $N \models \varphi(a)$, then there is $y \in N$ such that $N \models \psi(y, a)$, so there is $j \in \mathbb{N}$ such that $y \in N_j$. If $j \leq i$, then $N_j \subseteq N_i$, so $y \in N_i$ and by the induction assumption, $N_i \models \psi(y, a)$, so $N_i \models \varphi(a)$. If $i < j$, then by the induction assumption, $N_j \models \psi(y, a)$, so $N_j \models \varphi(a)$ and by Remark 3.16, $N_i \models \varphi(a)$.

□

At first sight, an elementary extension N of M can be much more complicated than M, so its geometric structure would not have to be a projective plane. But when we look closer, we can show that the structure of N is so rigid that M contains an isomorphic copy of the finite subgraphs of N. This rigidity allows us show that the elementary extensions of M preserve the dimension and the strong submodel relation, so they are not much more complicated than M.

Lemma 3.18. *If $B \subseteq N$ is finite, then there is $A \subseteq M$ such that $A \cong_{B \cap M} B$.*

Proof. Take $B = \{b_1, \ldots, b_n\} \subseteq N$, denote $B \cap M$ as X, and encode the relations in B with a formula ψ. By construction, $N \models \varphi(X)$ where

$$\varphi(X) := \exists y_1 \ldots \exists y_m \psi(y_1, \ldots, y_m, X)$$

and $m = n - |X|$, so $M \models \varphi(X)$ and there is $A = y_1 \ldots y_m X \subseteq M$ with the same relations as B, so $A \cong_X B$. □

Proposition 3.19. *If $A \subseteq M$, then $d_M(A) = d_N(A)$.*

Proof. By definition, $d_N(A) \leq d_M(A)$, so if towards contradiction there is $A \subseteq M$ such that $d_M(A) \neq d_N(A)$, then there is $A \subseteq B \subseteq N$ such that $\delta(B) = d_N(A) < d_M(A)$, so by Lemma 3.18, there is $A \subseteq B' \subseteq M$ such that $B' \cong_A B$, so $\delta(B') = \delta(B) < d_M(A)$ — a contradiction. □

Corollary 3.20. *If $A \leq M$, then $A \leq N$.*

Proof. The result follows from Proposition 1.23 and Proposition 3.19. □

The elementary extensions of M are so similar to M that they are also projective planes. Their geometric structures are then the same, so M is not an exceptional model, but rather part of a collection of models sharing many of the same properties.

Theorem 3.21. *Every N is a projective plane.*

Proof. Let $N \models \mathrm{Th}(M)$. The sentences

$$\varphi = \forall a \forall b (a \neq b \to \exists! c (aRc \wedge bRc))$$

and

$$\psi = \exists a_1 \dots \exists a_4 (\bigwedge_{1 \le i < j \le 4} a_i \ne a_j \wedge \bigwedge_{1 \le i < j < k \le 4} \neg \exists b (a_i R b \wedge a_j R b \wedge a_k R b))$$

encode the definition of a projective plane, so by Theorem 3.14, $M \models \varphi \wedge \psi$ and $\varphi \wedge \psi \in \mathrm{Th}(M)$, so $N \models \varphi \wedge \psi$ and N is a projective plane. $\square$

3.4 A First-Order Axiomatization of Th(M)

Before we keep showing the relation between M and its elementary extensions, we need to take a step back and rethink the definition of M, because there is a problem. The last condition of the definition of M, which allowed us to show that M is homogeneous, is not expressible in first-order logic and, even worse, does not carry to elementary extensions. Thus, nothing guarantees us that the elementary extensions of M are homogeneous. To show that they are, we characterize M with some other conditions which, even though are also not expressible in first-order logic, at least carry on to elementary extensions.

But finding these conditions M is tricky. We want to find conditions so specific that any model satisfying them is isomorphic to M, but so general that every elementary extension of M satisfies them. So to find them, we will remove all the graphs of M that do not provide any new information, and just keep the graphs that do. Thus, intuitively, we will be finding the skeleton of M.

Definition 3.22. A (possibly infinite) graph A is *strongly independent over* a finite graph B, denoted $A \downarrow_B$, if

$$d(X/B) = \delta(X/B) = |X| \qquad \forall X \subseteq A \text{ finite.}$$

Strongly independent graphs constitute then the skeleton of M and, to understand them better, we show a few of their properties.

Lemma 3.23. *If $A \downarrow_B$ and $B \le M$, then*

$$XB \le M \qquad \forall X \subseteq A \textit{ finite.}$$

Proof. The result follows from Lemma 3.8. $\square$

Lemma 3.24. *For any $AC \subseteq \ell_b$, if $A \downarrow_{Cb}$ and $C \downarrow_b$, then $AC \downarrow_b$.*

Proof. Let $X \subseteq AC$ finite and denote $X_A = X \cap A$ and $X_C = X \setminus A$. By Corollary 1.36 and Lemma 1.37,

$$d(X/b) = d(X_C/b) + d(X_A/X_C b) \ge |X_C| + d(X_A/Cb) = |X|,$$

and by Corollary 1.38 and Lemma 3.7,

$$d(X/b) \leq \sum_{x \in X} d(x/b) \leq |X|.$$

The same argument applies to $\delta(X/b)$, so $AC \downarrow_b$. □

As it turns out, the graph M is so rich that, for each of its strong submodels $X \leq M$, there is an infinite graph I strongly independent over X. There is then an infinite amount of information to add to any strong submodel. But is I unique? Well, no. In fact, there is a distinct I contained in each of the lines defined by a point of X.

Definition 3.25. The graph I_{Xb} is the maximal subset of ℓ_b such that $I_{Xb} \downarrow_{Xb}$.

Proposition 3.26. *If $Xb \leq M$, then I_{Xb} is infinite.*

Proof. We define by recursion on $n \in \mathbb{N}$ a collection of sets $\{I_n \leq M \mid n \in \mathbb{N}\}$ such that, for every $n \in \mathbb{N}$, $I_{n+1} = I_n a$ and $a \downarrow_{I_n}$.

- Let $I_0 = Xb$.
- Suppose that we have constructed I_n. Let $A = I_n a$ be a graph where a is only related to b. By construction, $A \in K^*$ and $I_n \leq A$, and since by the recursion assumption $I_n \leq M$, then there is a strong embedding $f : A \to M$ such that $f \restriction I_n = \mathrm{id}$, so

  $$d(f(a)/I_n) = \delta(f(a)/I_n) = 1$$

  and $f(a) \downarrow_{I_n}$. By Lemma 3.23, $I_n f(a) \leq M$, so let $I_{n+1} = I_n f(a)$.

By construction, the set $I = \bigcup_{n \in \mathbb{N}} I_n - Xb$ is an infinite subset of ℓ_b and, by Lemma 3.24, it is strongly independent over Xb, so I_{Xb} must be infinite. □

We are now ready to characterize M. The first three conditions remain the same, but we replace the fourth condition with the following new, three conditions. Every 0-simply algebraic extension has the maximum number of isomorphic copies, there are infinitely many unrelated points, and every line contains an infinite strongly minimal set over its defining point. In Lemma 3.13 and Proposition 3.26, we showed that M satisfies the second and third new conditions, so to show that M satisfies the characterization, we just need to show that M satisfies the first, new condition.

Lemma 3.27. *If C is 0-simply algebraic over B and $B \leq_C M$, then there is $F \subseteq B$ such that C is minimally 0-simply algebraic over F and*

$$\chi_M(C/F) = \mu(C/F).$$

Proof. Take $B, C \subseteq M$ such that C is 0-simply algebraic over B and $B \leq_C M$. By Proposition 2.8, there is a unique $F \subseteq B$ such that C is minimally 0-simply algebraic over F. We construct by recursion on $n \in \mathbb{N}$ a collection of graphs $\{D_n \subseteq M \mid n \leq \mu(C/F)\}$ such that $\chi_{D_n}(C/F) \geq n$.

- Let $D_0 = B$.
- Suppose that we have constructed D_n. By assumption, C is 0-simply algebraic over B, so by Remark 2.2, $B \leq BC$, and since $B \leq_C M$ and $B \subseteq D_n \subseteq M$, then $B \leq_C D_n$, so we can apply Lemma 2.20 with $B_0 = B, B_1 = BC, B_2 = D_n$ to get two cases:
    1. If $A = D_n \otimes_B C \in K^*$, then there is a strong embedding $f : A \to M$. The graph $D_{n+1} = f(A)$ contains a copy of C disjoint from D_n, so by the recursion hypothesis,
    $$\chi_{D_{n+1}}(C/F) = \chi_{D_n}(C/F) + 1 \geq n + 1.$$
    2. If $\chi_{D_n}(C/F) = \mu(C/F)$, then
    $$\mu(C/F) = \chi_{D_n}(C/F) \leq \chi_M(C/F) \leq \mu(C/F),$$
    so $\chi_M(C/F) = \mu(C/F)$.

If case 2 happens, we are done, and otherwise,

$$\mu(C/F) \leq \chi_{D_{\mu(C/F)}}(C/F) \leq \chi_M(C/F) \leq \mu(C/F),$$

so $\chi_M(C/F) = \mu(C/F)$ and we are done. □

After showing that M satisfies the characterization, we can go ahead and show that all K^*-universal-homogeneous graphs also satisfy it.

Theorem 3.28. *A countable graph N is K^*-universal-homogeneous if and only if it satisfies the following conditions.*

1. *Every $A \leq N$ is in K^*.*
2. *For every finite $A \subseteq N$, there is $B \in K^*$ such that $A \subseteq B \leq N$.*
3. *For every $A \in K^*$, there is a strong embedding $f : A \to N$.*
4. *If C is 0-simply algebraic over B and $B \leq_C N$, then there is $F \subseteq B$ such that C is minimally 0-simply algebraic over F and*
$$\chi_N(C/F) = \mu(C/F).$$
5. *There is an infinite subset $D \subseteq N$ such that*
$$d(X) = 2\,|X| \qquad \forall X \subseteq D \text{ finite.}$$

6. *For every $b \in N$, there is an infinite $I_b \subseteq N$ such that $I_b \downarrow_b$.*

Proof. By definition, M satisfies conditions $1, 2, 3$, by Lemma 3.27, condition 4, by Lemma 3.13, condition 5, and by Proposition 3.26, condition 6, so we just need to show that any countable graph N satisfying the conditions is K^*-universal-homogeneous. Let N be such a graph. By definition, N satisfies conditions $1, 2, 3$ of the definition of K^*-universal-homogeneous, so we just need to show that it satisfies condition 4 of the definition.

Take $B, C \in K^*$ such that $B \leq C$ and $B \leq N$. We find by induction on $|C \setminus B|$ a strong embedding of C into N fixing B.

- If $|C \setminus B| = 0$, then $C = B$, and since $B \leq N$, the identity map is a strong embedding of C into N fixing B.

- Suppose that $|C \setminus B| = n + 1$ and, for any $D \in K^*$ such that $B \leq D$ and $|D \setminus B| \leq n$, there is a strong embedding $f : D \to N$ such that $f \upharpoonright B = \text{id}$.

  If there is $D \in K^*$ such that $B \leq D \leq C$ and $B \subsetneq D \subsetneq C$, then by the induction hypothesis, since $B \leq N$ and $|D \setminus B| \leq n$, there is a strong embedding $f : D \to N$ such that $f \upharpoonright B = \text{id}$. We can find $C' \in K^*$ isomorphic to C such that $f(D) \subseteq C'$, so there is an isomorphism $g : C \to C'$ such that $f \subseteq g$. By construction, $f(D) \leq N$, $f(D) \leq C'$, and $|C' \setminus f(D)| \leq n$, so by the induction hypothesis, there is a strong embedding $h : C' \to N$ such that $h \upharpoonright f(D) = \text{id}$. The map $h \circ g$ is then a strong embedding of C into N fixing B, so we are done.

  We may therefore suppose that there is no such D. In particular, if there is no such D, then $\delta(D/B) > \delta(C/B)$ for every D such that $B \subsetneq D \subsetneq C$ since, otherwise, the D with minimal y would satisfy $B \leq D \leq C$. There are then three cases to consider:

  1. If $\delta(C/B) = 0$, then $\delta(D/B) > 0$ for every $B \subsetneq D \subsetneq C$, so C is 0-simply algebraic over B.
  2. If $\delta(C/B) = 1$, then $\delta(D/B) > 1$ for every $B \subsetneq D \subsetneq C$, so C is 1-simply algebraic over B and, by Proposition 2.10, $C = Ba$ and $r(a, B) = 1$.
  3. If $\delta(C/B) \geq 2$, then, since $\delta(a/B) \leq 2$ for any $a \in C - B$, we get that $C = Ba$ and $r(B, a) = 0$.

  We now find the strong embedding for each of the three cases.

  1. C is 0-simply algebraic over B.
     By Proposition 2.8, there is a unique $F \subseteq B$ such that C is minimally 0-simply algebraic over F. By assumption, $B \leq N$, so by Lemma 1.21, $B \leq_C N$. By condition 4,

$$\chi_N(C/F) = \mu(C/F) \geq 1,$$

so by Lemma 2.21, there is $C' \leq N$ isomorphic to C over B, so the isomorphism is the strong embedding of C into N fixing B.

2. $C = Ba$ and $r(a, B) = 1$.

   By assumption, $r(a, B) = 1$, so there is a unique $b \in B$ related to a. By condition 6, there is an infinite set I_b strongly independent over b, so we can take $X \subseteq I_b$ such that $|X| = d(B) + 1$. If we can find $c \in X$ such that $\delta(c/B) = 1$ and $Bc \leq N$, then $Ba \cong Bc$ and the isomorphism would be a strong embedding of C into N fixing B.

   By Lemma 3.7, for every $c \in X$,

$$d(c/B) \leq \delta(c/\overline{B}) \leq 1.$$

   If $d(c/B) \leq 0$ for every $c \in X$, then by Corollary 1.38,

$$\begin{aligned}
0 &\leq d(B - b/Xb) \\
&= d(XB) - d(Xb) \\
&\leq d(XB) - d(B) + d(B) - d(X/b) \\
&= d(X/B) + d(B) - |X| \\
&\leq \sum_{c \in X} d(c/B) + d(B) - d(B) - 1 \\
&= -1,
\end{aligned}$$

   so there is $c \in X$ such that $d(c/B) = 1$. Such a c cannot be in $\overline{B}$, so by Corollary 1.11 and Proposition 1.9,

$$1 = \delta(c/\overline{B}) \leq \delta(c/B) = \delta(c) - r(c, B) \leq 1,$$

   so $d(c/B) = \delta(c/B) = 1$ and, by Proposition 1.23, $Bc \leq N$.

3. $C = Ba$ and $r(a, B) = 0$.

   By condition 5, there is an infinite graph $D \subseteq N$ such that $d(X) = 2|X|$ for every finite $X \subseteq D$, so we can take $X \subseteq D$ such that $|X| = d(B) + 1$. If we can find $c \in X$ such that $\delta(c/B) = 2$ and $Bc \leq N$, then $Ba \cong Bc$ and the isomorphism would be a strong embedding of C into N fixing B.

   By Lemma 1.32 and Proposition 1.9, for every $c \in X$,

$$d(c/B) \leq \delta(c/\overline{B}) = \delta(c) - r(c, \overline{B}) = 2 - r(c, \overline{B}) \leq 2.$$

If $d(c/B) \leq 1$ for every $c \in X$, then by Corollary 1.38,

$$\begin{aligned} 0 &\leq d(B/X) \\ &= d(XB) - d(X) \\ &= d(XB) - d(B) + d(B) - d(X) \\ &= d(X/B) + d(B) - 2\,|X| \\ &\leq \sum_{c \in X} d(c/B) + |X| - 1 - 2\,|X| \\ &\leq |X| - |X| - 1 \\ &= -1, \end{aligned}$$

so there is $c \in X$ such that $d(c/B) = 2$. Such a c cannot be in $\overline{B}$, so by Corollary 1.11 and Proposition 1.9,

$$2 = \delta(c/\overline{B}) \leq \delta(c/B) = \delta(c) - r(c, B) \leq 2,$$

so $d(c/B) = \delta(c/B) = 2$ and, by Proposition 1.23, $Bc \leq N$.

□

Chapter 4

Classifying M

We are now ready classify M, showing that M is definable over a strongly minimal set, meaning that M is easy to define, and yet M does not admit a coordinate system over a division ring, meaning that its geometry is complicated.

4.1 Characterizing Definability

But first things first, we need to formalize what we mean by definability.

Definition 4.1. Let P be a structure. The *definable closure* of $A \subseteq P$ in P, denoted $\mathrm{dcl}_P(A)$, is the set of elements of P that are uniquely defined by a formula with parameters in A.

We give two immediate properties of the definable closure.

Remark 4.2 (Monotonicity). If $A \subseteq B$, then $\mathrm{dcl}(A) \subseteq \mathrm{dcl}(B)$.

Lemma 4.3 (Extensitivity). *$X \subseteq \mathrm{dcl}(X)$.*

Proof. For any $a \in X$, the formula $\varphi(x, a) := x = a$ has parameters in X and is uniquely satisfied by a, so $a \in \mathrm{dcl}(X)$. □

In Chapter 3, we showed that every elementary extension of M is a projective plane (Theorem 3.21), so the geometric structure of $Th(M)$ is quite rigid. But how rigid exactly? Well, rigid enough for us to be able to define all the points of any elementary extension of M from just a line and two points. Defining then the elementary extensions of M, and therefore M itself, with just a few parameters.

Theorem 4.4. *If $b \in N$ and $p, q \in N - \ell_b$ are distinct, then*

$$N \subseteq \mathrm{dcl}(\{p, q\} \cup \ell_b).$$

Proof. Let $a \in N$. By Theorem 3.21, N is a projective plane, so the points p, a lie on a unique line ℓ_{pa}, the points q, a lie on a unique line ℓ_{qa}, and the lines ℓ_{pa}, ℓ_{qa} intersect ℓ_b at unique points p', q'. The points p, p' lie on a unique line $\ell_{pp'}$, and since by construction they lie on ℓ_{pa}, then $\ell_{pp'} = \ell_{pa}$ and, similarly, $\ell_{qq'} = \ell_{qa}$. The lines $\ell_{pp'}$ and $\ell_{qq'}$ intersect at a unique point, and since $a \in \ell_{pa} \cap \ell_{qa} = \ell_{pp'} \cap \ell_{qq'}$, then $\ell_{pp'}$ and $\ell_{qq'}$ intersect at a. Thus,

$$\varphi_a(x, p, q, p', q') := \exists y \exists z (pRy \wedge p'Ry \wedge qRz \wedge q'Rz \wedge xRy \wedge xRz)$$

is a formula with parameters in $\{p, q, p', q'\}$ uniquely satisfied by a, so

$$a \in \operatorname{dcl}(p, q, p', q') \subseteq \operatorname{dcl}(\{p, q\} \cup \ell_b).$$

□

For most of our purposes, however, the definable closure will turn out to be too restrictive. The model M is homogeneous everywhere, so all of its points have many different copies, and finding a formula that uniquely defines a single point is usually too hard. Instead, we can approximate points by finding a formula that only defines them and finitely many other points.

Definition 4.5. Let P be a structure. The *algebraic closure* of $A \subseteq P$ in P, denoted $\operatorname{acl}_P(A)$, is the set of all elements $c \in P$ for which there is a formula $\varphi(x, \overline{a})$ with parameters $\overline{a} \in A$ such that $P \models \varphi(c, \overline{a})$ and $\varphi(P, \overline{a})$ is finite.

It follows immediately from the definition that the algebraic closure is monotonic and contains the definable closure.

Remark 4.6. If $A \subseteq B$, then $\operatorname{acl}(A) \subseteq \operatorname{acl}(B)$.

Remark 4.7. $\operatorname{dcl}(A) \subseteq \operatorname{acl}(A)$.

Furthermore, elementary extensions preserve the algebraic closure.

Lemma 4.8. *If $P \preccurlyeq Q$ and $A \subseteq P$, then* $\operatorname{acl}_P(A) = \operatorname{acl}_Q(A)$.

Proof. If $b \in \operatorname{acl}_P(A)$, then there is a formula $\varphi(x, \overline{a})$ with $\overline{a} \in A$ such that $P \models \varphi(b, \overline{a})$ and $\left|\varphi(P, \overline{b})\right| = n \in \mathbb{N}$, so $P \models \psi(b, \overline{a})$, where

$$\psi(b, \overline{a}) = \varphi(b, \overline{a}) \wedge \exists^n c(\varphi(c, \overline{a})),$$

and since $P \preccurlyeq Q$, then $Q \models \psi(b, \overline{a})$, so $b \in \operatorname{acl}_Q(A)$ and $\operatorname{acl}_P(A) \subseteq \operatorname{acl}_Q(A)$. Now, suppose towards contradiction that there is $b \in \operatorname{acl}_Q(A) \setminus \operatorname{acl}_P(A)$. Then there is a formula $\varphi(x, \overline{a})$ with $\overline{a} \in A$ such that $Q \models \varphi(b, \overline{a})$ and $\left|\varphi(Q, \overline{b})\right| = n \in \mathbb{N}$, so $Q \models \psi(\overline{a})$, where

$$\psi(\overline{a}) = \exists^n c(\varphi(c, \overline{a})).$$

By assumption, $P \preccurlyeq Q$, so $P \models \psi(\overline{a})$, but since $b \notin \operatorname{acl}_P(A)$, then there is $b' \in P$ such that $P \models \varphi(b', \overline{a})$ but $Q \not\models \varphi(b', \overline{a})$ — a contradiction. □

With the help of the following three lemmas, we can establish a relation between the algebraic closure and the dimension, connecting the model theory of the algebraic closure to the geometry of the dimension.

Lemma 4.9. *If* $Y \subseteq \operatorname{acl}(X)$*, then* $\operatorname{acl}(Y) \subseteq \operatorname{acl}(X)$*.*

Proof. Let $a \in \operatorname{acl}(Y)$ and let $\varphi(x, \overline{y})$ be the formula witnessing it with k-many realizations. Since $\overline{y} \subseteq Y \subseteq \operatorname{acl}(X)$, then for every $y_i \in \overline{y}$ there is a formula $\psi_i(x, \overline{x}_i)$ witnessing that $y_i \in \operatorname{acl}(X)$. The formula

$$\begin{aligned}\gamma(x, \overline{x}_1, \dots, \overline{x}_n) = &\exists z_1 \dots \exists z_n (\bigwedge_{i=1}^{n} \psi_i(z_i, \overline{x}_i) \wedge \varphi(x, z_1, \dots, z_n) \wedge \\ &\exists b_1 \dots \exists b_k (\bigwedge_{i=1}^{k} \varphi(b_i, z_1, \dots, z_n) \wedge \\ &\forall b(\varphi(b, z_1, \dots, z_n) \rightarrow \bigvee_{i=1}^{k} b = b_i)))\end{aligned}$$

has parameters in X, is realized by a, and only has finitely many realizations, so $a \in \operatorname{acl}(X)$. □

Lemma 4.10. *If* X *is finite, then* $\overline{X} \subseteq \operatorname{acl}_N(X)$*.*

Proof. Let $a \in \overline{X} = X \cup \{y_1, \dots, y_n\}$ and encode the relations in $\overline{X}$ with a formula ψ. If $a \in X$, then $a \in \operatorname{acl}_N(X)$ by Lemma 4.3, so suppose without loss of generality that $a = y_1$. The formula

$$\varphi(x, X) := \exists z_2 \dots \exists z_n \psi(x, z_2, \dots, z_n, X)$$

has parameters in X and is satisfied by a, so we just need to show that it only has finitely many realizations in N, and since $\overline{X}$ is finite, we just need to show that φ does not hold for any $b \in N \setminus \overline{X}$.

Suppose towards contradiction that φ holds for $b \in N \setminus \overline{X}$. Then there is $Y = bz_2 \dots z_n X \subseteq N$ isomorphic to $\overline{X}$ over X, and since $b \in Y$ and $b \notin \overline{X}$, then $\overline{X} \neq Y$. By Lemma 1.24, $\overline{X}$ is the unique, closed graph containing X, so Y cannot be closed, and

$$\delta(\overline{Y}) = d(Y) < \delta(Y) = \delta(\overline{X}) = d(X)$$

— a contradiction. □

Lemma 4.11. *If* C *is 0-simply algebraic over* $B \leq N$*, then* $C \subseteq acl_N(B)$*.*

Proof. Let $C = \{c_1, \dots, c_n\}$ and let ψ be a formula encoding the relations in CB. Take $a \in C$ and suppose without loss of generality that $a = c_1$. The formula

$$\varphi(x, B) := \exists y_2 \dots \exists y_n \psi(x, y_2, \dots, y_n, B),$$

has parameters in B and is realized by a, so we just need to show that $\varphi(N,B)$ is finite. Suppose towards contradiction that it is infinite. For each $z \in \varphi(N,B)$, there is

$$C_z = zy_2 \dots y_n \quad \text{such that} \quad C_z \cong_B C.$$

Every C_z is finite, so there are only finitely many $z' \in \varphi(N,B)$ such that $z' \neq z$ and $C_{z'} = C_z$, and since $\varphi(N,B)$ is infinite, there are infinitely many distinct C_zs. Take $z, z' \in \varphi(N,B)$ such that $C_z, C_{z'}$ are distinct. By assumption, $B \leq N$ and $C_z, C_{z'} \subseteq N$, so $B \leq (C_z B)(C_{z'} B)$ and, by Lemma 2.3, $C_z \cap C_{z'} = \emptyset$, so there are infinitely many, pairwise disjoint C_zs isomorphic to C over B. By Proposition 2.8, there is a unique $F \subseteq B$ such that C is minimally 0-simply algebraic over F, and since every C_z is isomorphic to C over B, then there are infinitely many, pairwise disjoint C_zs isomorphic to C over F — a contradiction since $\chi_N(C/F) \leq \mu(C/F)$. □

Proposition 4.12. *If $Xa \subseteq N$ is finite and $d(a/X) = 0$, then $a \in \operatorname{acl}_N(X)$.*

Proof. Let $Xa \subseteq N$ be a finite graph such that $d(a/X) = 0$. We construct by recursion on $n \in \mathbb{N}$ a sequence $\overline{X} = C_0 \subseteq \ldots \subseteq C_n = \overline{Xa}$ such that $C_i \leq N$ and C_{i+1} is 0-simply algebraic over C_i.

- Let $C_0 = \overline{X}$.
- Suppose that we have constructed C_i. Let

  $$\mathcal{C}_{i+1} = \{D \subseteq \overline{Xa} - C_i \mid \delta(D/C_i) = 0\}.$$

  By the recursion hypothesis, $\delta(C_j/C_{j-1}) = 0$ for every $1 \leq j \leq i$, so by Proposition 1.13,

  $$\begin{aligned} \delta(\overline{Xa}/C_i) &= \delta(\overline{Xa}/C_i) + \delta(C_i/C_{i-1}) \\ &= \delta(\overline{Xa}/C_{i-1}) \\ &= \ldots \\ &= \delta(\overline{Xa}/C_0). \end{aligned}$$

  Then, by Lemma 1.7 and Proposition 1.33,

  $$\delta(\overline{Xa} - C_i/C_i) = \delta(\overline{Xa}/C_i) = \delta(\overline{Xa}/C_0) = \delta(\overline{Xa}/\overline{X}) = d(a/X) = 0,$$

  so $\overline{Xa} - C_i \in \mathcal{C}_{i+1}$ and $\mathcal{C}_{i+1}$ is not empty. The graph $\overline{Xa}$ is finite, so $\mathcal{C}_{i+1}$ is finite and there is a subset minimal $D \in \mathcal{C}_{i+1}$. Let $C_{i+1} = DC_i$.

The sequence is as we wanted since:

1. The set $\overline{Xa}$ is finite, so it only has finitely many subsets and the sequence must be finite. As above, $\overline{Xa} - C_i \in \mathcal{C}_{i+1}$, so for the sequence to finish the last element contains $\overline{Xa} - C_{n-1}$, so

   $$C_n = (\overline{Xa} - C_{n-1})C_{n-1} = \overline{Xa}C_{n-1} = \overline{Xa}.$$

2. By definition, we have that $\overline{Xa} \leq N$, so applying recursively Remark 2.2 and Lemma 1.17, we get that $C_i \leq N$ for every $i \leq n$.

3. By construction, for every $0 \leq i < n$, $\delta(C_{i+1}/C_i) = 0$ and $\delta(D/C_i) > 0$ for every $C_i \subsetneq D \subsetneq C_{i+1}$, so C_{i+1} is 0-simply algebraic over C_i.

By Lemma 4.11, $C_{i+1} \subseteq \mathrm{acl}_N(C_i)$ for every $i < n$, so by Lemma 4.9, $\mathrm{acl}_N(C_{i+1}) \subseteq \mathrm{acl}_N(C_i)$. By Lemmas 4.10 and 4.9, $\mathrm{acl}_N(\overline{X}) \subseteq \mathrm{acl}_N(X)$, so

$$a \in \overline{Xa} = C_n \subseteq \mathrm{acl}_N(C_{n-1}) \subseteq \ldots \subseteq \mathrm{acl}_N(C_0) = \mathrm{acl}_N(\overline{X}) \subseteq \mathrm{acl}_N(X).$$

□

In the lines of M, the relation between the algebraic closure and the dimension is so strong that they are equivalent.

Corollary 4.13. *If $Xb \leq M$ and $Ya \subseteq \ell_b$ is finite, then*

$$d_M(a/YXb) = \begin{cases} 0 & a \in \mathrm{acl}_M(YX \cup \{b\}) \\ 1 & a \notin \mathrm{acl}_M(YX \cup \{b\}) \end{cases}$$

Proof. By Lemma 3.7, $d_M(a/YXb) \in \{0, 1\}$. If $a \notin \mathrm{acl}_M(YX \cup \{b\})$, then by Proposition 4.12, $d_M(a/YXb) = 1$. If $a \in \mathrm{acl}_M(YX \cup \{b\})$, then there can only be finitely many elements automorphic to a over YXb. By Proposition 3.26, there are infinitely many $c \in I_{Xb} \subseteq \ell_b$ such that $d_M(c/YXb) = |c| = 1$, and since by Proposition 3.10 they are all automorphic to each other over YXb, then $d(a/YXb) = 0$. □

4.2 Uniqueness of M

This relation between the algebraic closure and the dimension allows us to show that the elementary extensions of M are all isomorphic to M. We show it in three steps. First, we show that any large enough $M \preccurlyeq N$ is isomorphic to M, then we show that every N has a large enough $N \preccurlyeq N'$, and finally, we show that every $M \preccurlyeq N$ is isomorphic to M.

We start by showing the first step. But first, we need to formalize what we mean by a *large enough* elementary extension of M.

Definition 4.14. A structure P is *acl-unbounded* if

$$P \not\subseteq \mathrm{acl}(X) \qquad \forall X \subseteq P \text{ finite.}$$

With the definition behind us, we can show the first step.

Proposition 4.15. *If N is acl-unbounded, then $N \cong M$.*

Proof. By Corollary 3.5, we just need to show that N satisfies the characterization of Theorem 3.28.

1. Let $B \leq N$. By Lemma 3.18, there is a finite $A \subseteq M$ isomorphic to B, and since by the definition of M there is $\overline{A} \in K^*$ such that $A \subseteq \overline{A} \leq M$, then by Propositions 2.15 and 2.14, $B \in K^*$.

2. Let $B \subseteq N$ be finite. By Lemma 1.24, there is a closed graph $\overline{B} \subseteq N$ containing B, so by Proposition 1.23, $B \subseteq \overline{B} \leq N$.

3. For every $A \in K^*$, there is a strong embedding $f : A \to M$, and by Corollary 3.20, $f(A) \leq N$.

4. Let $B, C \subseteq N$ be finite graphs such that C is 0-simply algebraic over B and $B \leq_C N$. By Proposition 2.8, there is $F \subseteq B$ such that C is minimally 0-simply algebraic over F, so we just need to show that $\chi_N(C/F) = \mu(C/F)$. We can encode in first-order logic the sentence φ expressing that "if $C' \cong C$, $F' \cong F$, and $C' \cap F' = \emptyset$, then there are exactly $\mu(C/F)$ many pairwise disjoint, isomorphic copies of C over F". By Lemma 3.27, $M \models \varphi$, so $N \models \varphi$ and $\chi_N(C/F) = \mu(C/F)$.

5. By Proposition 3.19 and Lemma 3.13, there is $D \subseteq M \subseteq N$ infinite such that, for any finite $X \subseteq D$,

$$d_N(X) = d_M(X) = \delta(X) = 2\,|X|\,.$$

6. If we can show that for all $b \in N$ and $X \subseteq \ell_b$ finite, there is $a \in \ell_b$ such that $d(a/X) = 1$, then the proof follows by Lemmas 3.23 and 3.24.

   Suppose towards contradiction that $d(a/X) = 0$ for every $a \in \ell_b$. Let $p, q \in N \setminus \ell_b$ be distinct points. By Proposition 4.12, $\ell_b \subseteq \mathrm{acl}_N(X)$, so by Theorem 4.4 and Lemma 4.9,

$$N \subseteq \mathrm{dcl}_N(p, q, \ell_b) \subseteq \mathrm{acl}_N(p, q, \ell_b) \subseteq \mathrm{acl}_N(p, q, X)$$

   — a contradiction since N is acl-unbounded.

□

To show the second step, we introduce types.

Definition 4.16. Let P be a structure and $A \subseteq P$ be a set. A *type* p over A is a set of formulas $\varphi(x, \overline{a})$ with parameters $\overline{a} \in A$. An element $b \in P$ *realizes* p, denoted $b \models p$, if

$$P \models \varphi(b, \overline{a}) \qquad \forall \varphi(x, \overline{a}) \in p.$$

We write $P \models p$ when there is $b \in P$ realizing p.

A type is then a description of an element, and a realization of a type is an element matching the type's description. Types are a fundamental concept in model theory, and we will use them throughout the chapter to get a better understanding of the model theory of M. In particular, we can use them now to show that the last two steps of our proof, proving then that M is isomorphic to any countable $M \preccurlyeq N$.

Lemma 4.17. *Every N has a countable $N \preccurlyeq N'$ that is acl-unbounded.*

Proof. We build by recursion on $i \in \mathbb{N}$ a chain of countable elementary extensions $N = N_0 \preccurlyeq N_1 \preccurlyeq \dots$ such that, for every $i \in N$, $N_{i+1} \models p_i$ where

$$p_i = \{\neg\varphi(x, \overline{a}) \mid \overline{a} \in N_i \text{ and } |\varphi(N_i, \overline{a})| \in \mathbb{N}\}.$$

- Let $N_0 = N$.
- If we have constructed N_i, then by Löwenheim-Skolem's Theorem, there is a countable $N_i \preccurlyeq N_{i+1}$ such that $N_{i+1} \models p_i$.

The model $N' = \bigcup_{i \in \mathbb{N}} N_i$ is countable. By Lemma 3.17, $N_i \preccurlyeq N'$ for every $i \in \mathbb{N}$, so $N \preccurlyeq N'$ and we just need to show that N' is acl-unbounded. Suppose towards contradiction that it is not. Then there is a finite $A \subseteq N'$ such that $N' \subseteq \operatorname{acl}_{N'}(A)$. By construction, there are $i \in \mathbb{N}$ such that $A \subseteq N_i$ and $b_i \in N_{i+1}$ such that $b_i \models p_i$. By Lemma 4.8,

$$b_i \in N_{i+1} \subseteq N' \subseteq \operatorname{acl}_{N'}(A) = \operatorname{acl}_{N_i}(A),$$

so there is a formula $\varphi(x, \overline{a})$ such that $|\varphi(N_i, \overline{a})| \in \mathbb{N}$ and $N_i \models \varphi(b_i, \overline{a})$, so $N_{i+1} \models \varphi(b_i, \overline{a})$, but since $b_i \models p_i$, then $N_{i+1} \models \neg\varphi(b_i, \overline{a})$ — a contradiction. □

Theorem 4.18. *Every N is isomorphic to M.*

Proof. By Proposition 4.15, we just need to show that N is acl-unbounded. Suppose towards contradiction that it is not. Then there is a finite set $X \subseteq N$ such that $N \subseteq \operatorname{acl}(X)$. Take $b \in M$ and $p, q \in M \setminus \ell_b$ distinct. By Theorem 4.4, $N \subseteq \operatorname{dcl}(p, q, \ell_b)$, so $X \subseteq \operatorname{dcl}(p, q, \ell_b)$, and since X is finite, there is a finite $Y \subseteq \ell_b$ such that $X \subseteq \operatorname{dcl}(p, q, Y)$. By the maximality of I_{bpq}, for every $y \in Y$ there is a finite $Z_y \subseteq I_{bpq}$ such that $d(y/Z_y bpq) = 0$. Let $Z = \bigcup_{y \in Y} Z_y$. By Lemma 4.9 and Proposition 4.12,

$$N \subseteq \operatorname{acl}(X) \subseteq \operatorname{acl}(p, q, Y) \subseteq \operatorname{acl}(b, p, q, Z).$$

By Proposition 3.26, the set I_{bpq} is infinite in M, and since by Proposition 3.19 elementary extensions preserve the dimension, then I_{bpq} is also infinite in N. The set $Z \subseteq I_{bpq}$ is finite, so there is $a \in I_{bpq}$ such that $d(a/Zbpq) = 1$. By Lemma 4.17 and Proposition 4.15, there is a countable $N \preccurlyeq N'$ isomorphic to M, so Corollary 4.13 applies to N' and by Lemma 4.8,

$$a \notin \operatorname{acl}_{N'}(b, p, q, Z) = \operatorname{acl}_N(b, p, q, Z) \supseteq N$$

— a contradiction since $a \in N$. □

4.3 Saturation of M

We can also use types to measure how rich M is. As we introduced above, types are descriptions of elements, so we can measure how rich M is by trying to find describable elements that are not in M. These elements would, intuitively, constitute *holes* in M, and if M does not have any holes, any describable missing element, then we will say that M is rich enough.

Definition 4.19. A type p in P over A is *consistent* if there is $P \preccurlyeq P'$ such that $P' \models p$, and *complete* if, for all formulas $\varphi(x, \overline{a})$ with parameters $\overline{a} \in A$, either $\varphi(x, \overline{a}) \in p$ or $\neg\varphi(x, \overline{a}) \in p$. The *type space* of A in P, denoted $S(A)$, is the set of all consistent, complete types in P over A.

A structure P is *ω-saturated* if for all finite $A \subseteq P$ and $p \in S(A)$, $P \models p$.

We cannot expect M to realize every possible type, of course, since there are many that do not make sense. So instead, we define consistent types to be the descriptions that make sense, and we say that M is rich enough, so ω-saturated, if there is an element satisfying each complete, sensible description over finitely many parameters.

At first sight, it may seem hard to prove that M is ω-saturated, but we have already done all the hard work. We have shown that M is isomorphic to all of its elementary extensions. So now we just need to show a basic lemma about the monotonicity of consistent, complete types, and combine both results.

Lemma 4.20. *If $p \in S(A)$ and $A \subseteq B$, then there is $q \in S(B)$ such that*

$$p = \{\varphi(x, \overline{b}) \in q \mid \overline{b} \in A\}.$$

Proof. Let P be a structure such that $A \subseteq B \subseteq P$ and let $p \in S(A)$. Then there is $P \preccurlyeq Q$ and $c \in Q$ such that $c \models p$. Let

$$q = p \cup \{\varphi(x, \overline{b}) \mid \overline{b} \in B \text{ and } Q \models \varphi(c, \overline{b})\}.$$

Every $\varphi(x, \overline{a}) \in p$ has parameters in $A \subseteq B$, and since $Q \models q$, then q is a consistent type over B in P. For every formula $\varphi(x, \overline{b})$ with $\overline{b} \in B$, either $Q \models \varphi(c, \overline{b})$ or $Q \models \neg\varphi(c, \overline{b})$, and since $\neg\varphi(x, \overline{b})$ is a formula with parameters in B, then either $\varphi(x, \overline{b}) \in q$ or $\neg\varphi(x, \overline{b}) \in q$, so q is complete and $q \in S(B)$. The type p is complete, so every $\varphi(x, \overline{b}) \in q$ with $\overline{b} \in A$ is in p — as required. □

Theorem 4.21. *M is ω-saturated.*

Proof. Let $A \subseteq M$ be finite and $p \in S(A)$. If f is a function, we denote

$$f(p) = \{\varphi(x, f(\overline{a})) \mid \varphi(x, \overline{a}) \in p\}.$$

By Lemma 4.20, there is $q \in S(\overline{A})$ such that $p \subseteq q$. There is then a countable $M \preccurlyeq N$ such that $N \models q$, so there is some $b \in N$ such that $b \models q$. By Theorem 4.18, $M \cong N$, so there is an isomorphism $f : N \to M$, and since isomorphisms preserve truth, then

$$f(b) \models f(q).$$

By Corollary 3.20, $\overline{A} \leq N$, so $B = f(\overline{A}) \leq M$, and $f^{-1} \upharpoonright B$ is an isomorphism from $B \leq M$ to $\overline{A} \leq M$. By Corollary 3.4, there is an automorphism g of M such that $f^{-1} \upharpoonright B \subseteq g$, and since automorphisms preserve truth, then

$$g(f(b)) \models g(f(q)).$$

By construction, $q \in S(\overline{A})$ and $f^{-1} \upharpoonright B \subseteq g$, so $g(f(q)) = q$, and since $g(f(b)) \in M$, then $M \models q$, so $M \models p$. □

4.4 M as an Almost Strongly Minimal Model

We are now ready to show one of the main results of the thesis, showing that M is almost strongly minimal. In the introduction, we said that a model is almost strongly minimal when we can define all of its elements with just a few parameters, and in Theorem 4.4, we showed that we can define all the elements of M using a line and two distinct points outside the line, so if we can show that the lines of M are "small", then we will have shown that M is almost strongly minimal. But what does small mean?

Definition 4.22. Let P be a structure. A set $S \subseteq P$ is *minimal* if it is definable by a formula $\varphi(x, \overline{a})$ and all of its definable subsets $D \subseteq S$ are either finite or cofinite ($S \setminus D$ is finite). A minimal set $S \subseteq P$ is *strongly minimal* if $\varphi(Q, \overline{a})$ is minimal for every $P \preccurlyeq Q$.

By *small* we then mean strongly minimal — a set that may be infinite, but behaves like a finite set in terms of definability. Our challenge is then to show that the lines of M are strongly minimal, which we show in three steps.

Lemma 4.23. *If $D \subseteq M$ is definable by a formula $\varphi(x, X)$ with parameters $X \subseteq M$ and $D \subseteq \operatorname{acl}(X)$, then D is finite.*

Proof. Let $\{\varphi_i(x, X) \mid i \in \mathbb{N}\}$ be the set of formulas witnessing that an element of D is algebraic over X. Suppose towards contradiction that $\varphi(x, X)$ is not equivalent to $\bigvee_{i \in I} \varphi_i(x, X)$ for any finite $I \subseteq \mathbb{N}$. The type

$$p = \{\varphi(x, X)\} \cup \{\neg\varphi_i(x, X) \mid i \in \mathbb{N}\}$$

is then consistent, so by Theorem 4.21, $M \models p$ and there is some $a \in D$ such that $a \notin \operatorname{acl}(X)$ — a contradiction. Thus, there is a finite $I \subseteq \mathbb{N}$ such that $\varphi(x, X)$ is equivalent to $\bigvee_{i \in I} \varphi_i(x, X)$, and since each φ_i only has finitely many realizations, then so does φ and D is finite. □

Proposition 4.24. *The lines of M are minimal.*

Proof. Let $b \in M$. The line ℓ_b is definable by the formula $\psi(x, b) = xRb$, so to show that ℓ_b is minimal we just need to show that any $D \subseteq \ell_b$ definable by a formula $\varphi(x, X)$ with parameters $X \subseteq M$ is either finite or cofinite.

By Proposition 3.10, all the points $a \in \ell_b$ such that $d(a/Xb) = 1$ are automorphic to each other over Xb, so they are all in either D or $\ell_b \setminus D$. If they are all in $\ell_b \setminus D$, then $d(a/Xb) = 0$ for every $a \in D$, so by Proposition 4.12, $D \subseteq \operatorname{acl}(X, b)$, and by Lemma 4.23, D is finite. Similarly, if they are all in D, then $\ell_b \setminus D$ is finite. □

Theorem 4.25. *The lines of M are strongly minimal.*

Proof. Suppose towards contradiction that there is $b \in M$ such that ℓ_b is not strongly minimal. Then there exist a set $X \subseteq \ell_b^M$, definable by a formula $\varphi(x, \overline{a})$, and an elementary extension $M \preccurlyeq N$ such that the set $Y = \varphi(N, \overline{a})$ is not minimal. Since Y is not minimal, there is a formula $\psi(x, \overline{c})$ such that the set $Y \cap \psi(N, \overline{c})$ is not finite or cofinite, so the sets $Y \cap \psi(N, \overline{c})$ and $Y \cap \neg\psi(N, \overline{c})$ are both infinite. Thus, for every $n \in \mathbb{N}$, the formulas

$$\gamma_n(\overline{c}, \overline{a}) = \exists^{\geq n} y(\varphi(y, \overline{a}) \wedge \psi(y, \overline{c})),$$
$$\gamma'_n(\overline{c}, \overline{a}) = \exists^{\geq n} y(\varphi(y, \overline{a}) \wedge \neg\psi(y, \overline{c}))$$

hold in N. All these formulas are in $\operatorname{tp}(\overline{c}/\overline{a})$, so by Theorem 4.21, there is $\overline{d} \in M$ such that, for every $n \in \mathbb{N}$,

$$M \models \gamma(\overline{d}, \overline{a}) \wedge \gamma'(\overline{d}, \overline{a}),$$

hold in N. These formulas have parameters in $\overline{a}$, so by Theorem 4.21, there is $\overline{d} \in M$ such that $M \models \gamma(\overline{d}, \overline{a})$ and $M \models \gamma'(\overline{d}, \overline{a})$. Thus, for every $n \in \mathbb{N}$,

$$\left|X \cap \psi(M, \overline{d})\right| \geq n \qquad \text{and} \qquad \left|X \cap \neg\psi(M, \overline{d})\right| \geq n,$$

so $\left|X \cap \psi(M, \overline{d})\right| \geq n$ and $\left|X \cap \neg\psi(M, \overline{d})\right| \geq n$ for every $n \in \mathbb{N}$ and X is not minimal in M — contradicting Proposition 4.24. □

We now define formally what an almost strongly minimal model is, and we show that M is almost strongly minimal.

Definition 4.26. Let P be a structure and $a, b \in P$ be tuples. The *complete type* of a over b in P,

$$\operatorname{tp}(a/b) = \{\varphi(x, b) \mid P \models \varphi(a, b)\},$$

is *isolated* if there is a formula $\varphi(x, b) \in \operatorname{tp}(a/b)$ such that, for all formulas $\psi(x, b) \in \operatorname{tp}(a/b)$,

$$P \models \forall x(\varphi(x, b) \rightarrow \psi(x, b)).$$

A structure P is *almost strongly minimal* if there is a strongly minimal set $S \subseteq P$ such that $P \subseteq \operatorname{acl}(S)$ and the complete type of the parameters used to define S is isolated.

The complete type of a over b is then the most complete description that b can make of a, so how b "sees" a, and it is isolated when a single formula captures it, so when it is simple. Thus, the description we gave above of an almost strongly minimal set was incomplete. We do not just require a strongly minimal set to approximate M, we also require the strongly minimal set approximating M to be "simple", to be definable by parameters that form an isolated type.

We then just need to find an isolated type defining M.

Lemma 4.27. *There are $b \in M$ and $p, q \in M \setminus \ell_b$ distinct such that their complete type $tp(bpq/\emptyset)$ is isolated.*

Proof. Let $bpq \subseteq M$ be a graph such that $r(b, pq) = 0$ and $d(bpq)$ is minimal among all such graphs. Let φ be a formula encoding the relations in bpq and let ψ be a formula encoding the relations in $\overline{bpq}$. Denote $|\overline{bpq}|$ by n and let γ be a formula such that

$$\gamma(x, y, z) := \varphi(x, y, z) \wedge \exists v_1 \dots \exists v_n \psi(x, y, z, v_1, \dots, v_n).$$

Let $\theta \in \text{tp}(bpq/\emptyset)$ and let $x, y, z \in M$ be such that $M \models \gamma(x, y, z)$. To show that $\text{tp}(bpq/\emptyset)$ is isolated and that γ witnesses it, we need to show that $M \models \theta(x, y, z)$.

By assumption, $M \models \gamma(x, y, z)$, so there are $v_1, \dots, v_n \in M$ such that $M \models \psi(x, y, z, v_1, \dots, v_n)$, and thus $\delta(xyzv_1 \dots v_n) = \delta(\overline{bpq})$. By construction the dimension $d(bpq)$ is minimal among the graphs that satisfy φ, and since $M \models \varphi(x, y, z)$, then there cannot be an extension of xyz with a lower predimension than $\overline{bpq}$, so

$$\overline{xyz} = xyzv_1 \dots v_n.$$

Both the graphs $\overline{xyz}$ and $\overline{bpq}$ satisfy ψ, so they are isomorphic to each other and by Corollary 3.4, there is an automorphism $f : M \to M$ such that

$$f(\overline{bpq}) = \overline{xyz}, \quad f(b) = x, \quad f(p) = y, \quad f(q) = z.$$

By assumption, $\theta \in \text{tp}(bpq/\emptyset)$, so $M \models \theta(b, p, q)$, and since automorphisms preserve truth, then $M \models \theta(f(b), f(p), f(q))$, so $M \models \theta(x, y, z)$. □

It then follows that M is almost strongly minimal.

Theorem 4.28. *M is almost strongly minimal.*

Proof. By Lemma 4.27, there are $b \in M$ and $p, q \in M \setminus \ell_b$ distinct such that $\text{tp}(bpq/\emptyset)$ is isolated. The set $\{p, q\} \cup \ell_b$ is definable by the formula

$$\varphi(x, b, p, q) = (xRb) \vee (x = p) \vee (x = q).$$

By Theorem 4.25, ℓ_b is strongly minimal, and since a strongly minimal set extended by a finite set is still strongly minimal, then the set $\{p, q\} \cup \ell_b$ is strongly minimal. But by Theorem 4.4,

$$M \subseteq \operatorname{dcl}(\{p, q\} \cup \ell_b) \subseteq \operatorname{acl}(\{p, q\} \cup \ell_b),$$

so M is almost strongly minimal. □

4.5 M as a non-Desarguesian Projective Plane

Finally, we show that M is non-Desarguesian, completing our construction.

Definition 4.29. A projective plane is *Desarguesian* if, for any two triangles ABC and $A'B'C'$, there is a line ℓ such that

$$AA' \cap BB' \cap CC' \neq \emptyset \iff AB \cap A'B', BC \cap B'C', CA \cap C'A' \in \ell.$$

We only include the definition of a Desarguesian projective plane for completeness. What matters to us is the well-known fact that Desarguesian projective planes have coordinates over a division ring. Proving this fact, however, would involve a number of pages on projective geometry, and since the focus of this thesis is not projective geometry, we skip it and instead cite Hilbert's proof.

Theorem 4.30. *If P is a Desarguesian affine plane and $\ell \subseteq P$ is a line, then there is a definable binary operation $*$ such that $(\ell, *)$ is a group.*

Proof. See Sections 24 and 25 of Hilbert [2]. □

Corollary 4.31. *If P is a Desarguesian projective plane, $\ell \subseteq P$ is a line, and $p \in \ell$ is a point, then there is a definable binary operation $*$ such that $(\ell \setminus \{p\}, *)$ is a group.*

Proof. Let P be a Desarguesian projective plane, $\ell \subseteq P$ be a line, and $p \in \ell$ be a point. Let $\ell' \subseteq P$ be a line such that $\ell \cap \ell' = \{p\}$. A Desarguesian projective plane without a line is a Desarguesian affine plane, so $P \setminus \ell'$ is a Desarguesian affine plane, and since $\ell \setminus \{p\}$ is a line in $P \setminus \ell'$, then by Theorem 4.30, there is a definable binary operation $*$ such that $(\ell \setminus \{p\}, *)$ is a group. □

We can then show that M is a non-Desarguesian projective plane by showing, with the help of the following lemma, that it is not possible to define a group over any of its lines without a point.

Lemma 4.32. *Let $A_1, \ldots, A_n, X \in K^*$. If $r(A_i - A_i, A_j - A_i) = 0$ and $A_i \cap X = \emptyset$ for every $1 \leq i, j \leq n$, then*

$$\delta\left(\bigcup_{1 \leq i \leq n} A_i \,/\, X\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \delta(\bigcap_{j=i_1}^{i_k} A_j / X).$$

Proof. By induction on $n \in \mathbb{N}$.

- If $n = 2$, then

$$
\begin{aligned}
\delta(A_1A_2) &= 2\,|A_1A_2| - e(A_1A_2) \\
&= 2\,|A_1| + 2\,|A_2| - 2\,|A_1 \cap A_2| - e(A_1) - e(A_2) + e(A_1 \cap A_2) \\
&\quad - r(A_1 - A_2, A_2 - A_1) \\
&= \delta(A_1) + \delta(A_2) - \delta(A_1 \cap A_2) - r(A_1 - A_2, A_2 - A_1) \\
&= \delta(A_1) + \delta(A_2) - \delta(A_1 \cap A_2),
\end{aligned}
$$

  and by Proposition 1.9,

$$
\begin{aligned}
\delta(A_1A_2/X) &= \delta(A_1A_2) - r(A_1A_2, X) \\
&= \delta(A_1) + \delta(A_2) - \delta(A_1 \cap A_2) - r(A_1, X) - r(A_2, X) + \\
&\quad r(A_1 \cap A_2, X) \\
&= \delta(A_1/X) + \delta(A_2/X) - \delta(A_1 \cap A_2/X).
\end{aligned}
$$

- If $n \geq 3$ and the formula holds for every union of $n-1$ sets, then as above,

$$
\begin{aligned}
\delta\left(\bigcup_{i=1}^{n} A_i/X\right) &= \delta(A_n/X) + \delta\left(\bigcup_{i=1}^{n-1} A_i/X\right) - \delta\left(A_n \cap \bigcup_{i=1}^{n-1} A_i/X\right) \\
&= \delta(A_n/X) + \delta\left(\bigcup_{i=1}^{n-1} A_i/X\right) - \delta\left(\bigcup_{i=1}^{n-1} (A_n \cap A_i)/X\right).
\end{aligned}
$$

  By the induction hypothesis,

$$
\delta\left(\bigcup_{i=1}^{n-1} A_i/X\right) = \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \delta(\bigcap_{j=i_1}^{i_k} A_j/X)
$$

  and

$$
\delta\left(\bigcup_{i=1}^{n-1} (A_n \cap A_i)/X\right) = \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \delta(A_n \cap \bigcap_{j=i_1}^{i_k} A_j/X).
$$

  Every subset of $\{A_1, \ldots, A_n\}$ either contains A_n, contributing to the second sum, or does not contain A_n, contributing to the first sum.

Thus,

$$\delta\left(\bigcup_{i=1}^{n} A_i/X\right) = \delta(A_n/X) + \sum_{k=1}^{n-1}(-1)^{k-1} \sum_{1\le i_1<\cdots<i_k\le n-1} \delta(\bigcap_{j=i_1}^{i_k} A_j/X)$$
$$- \sum_{k=1}^{n-1}(-1)^{k-1} \sum_{1\le i_1<\cdots<i_k\le n-1} \delta(A_n \cap \bigcap_{j=i_1}^{i_k} A_j/X)$$
$$= \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\cdots<i_k\le n} \delta(\bigcap_{j=i_1}^{i_k} A_j/X).$$

□

Theorem 4.33. *M is a non-Desarguesian projective plane.*

Proof. By Theorem 3.14, M is a projective plane, so suppose towards contradiction that M is Desarguesian. Let $b \in M$, let $p \in \ell_b$, and let $G = \ell_b \setminus \{p\}$. By Corollary 4.31, there is a definable binary operation $*$ such that $(G, *)$ is a group, so there is a formula φ with parameters $X \subseteq M$ such that, for all $x, y, z \in G$,

$$x * y = z \iff M \models \varphi(x, y, z, X).$$

We may assume without loss of generality that $b \in X$. By Proposition 3.26, the set $I_{\overline{X}}$ is infinite, so there are three distinct points $a_1, a_2, b_1 \in I_{\overline{X}} \setminus \{p\}$, and since $(G, *)$ is a group, then the points

$$a_3 = a_1 * a_2, \qquad b_2 = a_1 * b_1, \qquad b_3 = a_2^{-1} * b_1$$

are well-defined points of G. Let

$$A_1 = \{a_1, a_2, a_3\} \qquad\qquad A_3 = \{a_3, b_2, b_3\}$$
$$A_2 = \{a_2, b_1, b_3\} \qquad\qquad A_4 = \{a_1, b_1, b_2\}$$

and let $E = \bigcup_{1\le i\le 4} \overline{A_i X}$. By Lemma 1.32 and Corollary 1.38,

$$\delta(E/\overline{X}) \ge d(E/X) \ge d(a_1a_2b_1/X) = 3.$$

We now show, for a contradiction, that $\delta(E/\overline{X}) \le 2$. The proof is long, so we show it using different claims.

Claim 1. For any $i \in \{1, 2, 3, 4\}$,

$$\delta(\overline{A_i X}/\overline{X}) = 2.$$

Proof. Let $i \in \{1, 2, 3, 4\}$. By construction, any $S \subseteq A_i$ such that $|S| = 2$ is strongly independent over $\overline{X}$ and uniquely defines the remaining point. Denote $A_i = \{c_1, c_2, c_3\}$. By Lemmas 4.10 and 4.9,

$$c_3 \in \mathrm{dcl}(c_1, c_2, \overline{X}) \subseteq \mathrm{acl}(c_1, c_2, \overline{X}) \subseteq \mathrm{acl}(c_1, c_2, X),$$

so by Proposition 1.33 and Corollaries 1.36 and 4.13,

$$\delta(\overline{A_i X}/\overline{X}) = d(A_i/X) = d(c_1 c_2/X) + d(c_3/c_1 c_2 X) = 2.$$

□

Claim 2. For any $i, j \in \{1, 2, 3, 4\}$ distinct,

$$\delta(\overline{A_i X} \cap \overline{A_j X}/\overline{X}) = 1.$$

Proof. Let $i, j \in \{1, 2, 3, 4\}$ distinct. By construction, $A_i \cap A_j$ is a singleton strongly independent over $\overline{X}$, so by Corollary 1.35 and Claim 1,

$$1 = d(A_i \cap A_j/\overline{X}) \leq d(\overline{A_i X} \cap \overline{A_j X}/\overline{X}) \leq d(\overline{A_i X}/\overline{X}) = 2.$$

Suppose towards contradiction that $d(\overline{A_i X} \cap \overline{A_j X}/\overline{X}) = 2$. Then by Corollary 1.36 and Claim 1,

$$d(\overline{A_i X}/\overline{A_i X} \cap \overline{A_j X}) = d(\overline{A_i X}/\overline{X}) + d(\overline{A_i X} \cap \overline{A_j X}/\overline{X}) = 0,$$

so by Proposition 4.12,

$$A_i \subseteq \overline{A_i X} \subseteq \operatorname{acl}(\overline{A_i X} \cap \overline{A_j X}) \subseteq \operatorname{acl}(\overline{A_j X}).$$

But by construction there is $c \in A_i$ strongly independent over $\overline{A_j X}$, so by Corollary 4.13, $c \notin \operatorname{acl}(\overline{A_j X})$ — a contradiction. Thus, by Corollaries 1.27 and 1.34,

$$\delta(\overline{A_i X} \cap \overline{A_j X}/\overline{X}) = d(\overline{A_i X} \cap \overline{A_j X}/\overline{X}) = 1.$$

□

Claim 3. For any $i, j, k \in \{1, 2, 3, 4\}$ distinct,

$$\delta(\overline{A_i X} \cap \overline{A_j X} \cap \overline{A_k X}/\overline{X}) = 0.$$

Proof. Let $i, j, k \in \{1, 2, 3, 4\}$ distinct. By Claim 3,

$$0 \leq d(\overline{A_i X} \cap \overline{A_j X} \cap \overline{A_k X}/\overline{X}) \leq d(\overline{A_i X} \cap \overline{A_j X}/\overline{X}) = 1.$$

Suppose towards contradiction that $d(\overline{A_i X} \cap \overline{A_j X} \cap \overline{A_k X}/\overline{X}) = 1$. Then, as in the proof of Claim 2, $A_i \cap A_j \subseteq \operatorname{acl}(\overline{A_k X})$. But by construction $A_i \cap A_j$ is a singleton strongly independent over $\overline{A_k X}$, so by Corollary 4.13, $A_i \cap A_j \nsubseteq \operatorname{acl}(\overline{A_k X})$ — a contradiction. Thus, by Corollaries 1.27 and 1.34,

$$\delta(\overline{A_i X} \cap \overline{A_j X} \cap \overline{A_k X}/\overline{X}) = d(\overline{A_i X} \cap \overline{A_j X} \cap \overline{A_k X}/\overline{X}) = 0.$$

□

Claim 4.
$$\delta(\bigcap_{1\le i\le 4} \overline{A_iX}/\overline{X}) = 0.$$

Proof. By Claim 3,

$$0 \le d(\bigcap_{1\le i\le 4} \overline{A_iX}/\overline{X}) \le d(\bigcap_{1\le i\le 3} \overline{A_iX}/\overline{X}) = 0,$$

so by Corollary 1.34,

$$\delta(\bigcap_{1\le i\le 4} \overline{A_iX}/\overline{X}) = d(\bigcap_{1\le i\le 4} \overline{A_iX}/\overline{X}) = 0.$$

□

Claim 5. $r(\overline{A_iX} - \overline{A_jX}, \overline{A_jX} - \overline{A_iX}) = 0.$

Proof. By Lemma 4.10,

$$\overline{A_iX}\,\overline{A_jX} \subseteq \mathrm{acl}(A_iA_jX) \subseteq \mathrm{acl}(\{a_1, a_2, b_1\} \cup X),$$

so by Corollary 4.13 and Proposition 1.13,

$$d(\overline{A_iX}\,\overline{A_jX}/\overline{X}) = d(a_1a_2b_1/\overline{X}) - d(\overline{A_iX}\,\overline{A_jX}/a_1a_2b_1\overline{X}) = 3 + 0 = 3.$$

Thus, by Proposition 1.13, Lemma 1.32, and Claims 1 and 2,

$$\begin{aligned}\delta(\overline{A_iX} - \overline{A_jX}/\overline{A_jX}) &= \delta(\overline{A_iX}\,\overline{A_jX}/\overline{X}) - \delta(\overline{A_jX}/\overline{X})\\ &\ge d(\overline{A_iX}\,\overline{A_jX}/\overline{X}) - 2\\ &= 3 - 2\\ &= 1,\end{aligned}$$

and by Corollary 1.11,

$$\begin{aligned}1 &\le \delta(\overline{A_iX} - \overline{A_jX}/\overline{A_jX})\\ &\le \delta(\overline{A_iX} - \overline{A_jX}/\overline{A_iX} \cap \overline{A_jX})\\ &= \delta(\overline{A_iX}/\overline{X}) - \delta(\overline{A_iX} \cap \overline{A_jX}/\overline{X})\\ &= 2 - 1\\ &= 1,\end{aligned}$$

so
$$\delta(\bigcap_{1\le i\le 4} \overline{A_iX}/\overline{X}) = 0.$$

□

Now, Lemma 4.32 applies to $\delta(E - \overline{X}/\overline{X})$ by Claim 5, so by Lemma 1.7 and Claims 1-4,

$$\begin{aligned}
\delta(E/\overline{X}) &= \delta(E - \overline{X}/\overline{X}) \\
&= \sum_{k=1}^{4}(-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq 4} \delta(\bigcap_{j=i_1}^{i_k} \overline{A_j X} - \overline{X}/\overline{X}) \\
&= 4 \cdot \delta(\overline{A_1 X}/\overline{X}) - 6 \cdot \delta(\overline{A_1 X} \cap \overline{A_2 X}/\overline{X}) \\
&= 4 \cdot 2 - 6 \cdot 1 \\
&= 2
\end{aligned}$$

— a contradiction. □

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