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Trabajo Fin de Grado

Projective and Plane Curves: A relation between algebra, topology, and complex analysis

Author: Daniel Díaz Quílez
Tutor: Alfonso Zamora Saiz

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Autor: Daniel Díaz Quílez
Tutor: Alfonso Zamora Saiz
Departamento de Matemática Aplicada a las TIC
ETSI Informáticos
Universidad Politécnica de Madrid

Resumen

Los ceros de un polinomio sobre un cuerpo algebraic revelan relaciones entre el álgebra, la topología, y el análisis complejo. Son curvas algebraicas y, en este trabajo de fin de grado, las estudiamos sobre $\mathbb{C}^2$ (llamadas curvas afines) y sobre $\mathbb{P}^2_{\mathbb{C}}$ (llamadas curvas proyectivas), haciendo especial énfasis en las curvas planas (curvas proyectivas sin singularidades).

Empezamos estudiando los puntos de intersección entre dos curvas proyectivas con el teorema de Bézout y las relaciones de equivalencia entre las curvas cónicas proyectivas irreducibles y las curvas cúbicas proyectivas irreducibles bajo coordenadas proyectivas. Después nos centramos en las curvas planas y estudiamos su estructura topológica con el teorema del género grado y su relación con las superficies de Riemann con el teorema de Riemann-Roch.

Abstract

The zeros of a polynomial over an algebraic field reveal relations between algebra, topology, and complex analysis. They are algebraic curves and, in this end-of-degree project, we study them over $\mathbb{C}^2$ (called affine curves) and over $\mathbb{P}^2_{\mathbb{C}}$ (called projective curves), paying especial attention to plane curves (projective curves without singularities).

We start by studying the intersections between projective curves with Bézout's theorem and the equivalence relations between irreducible projective conics and irreducible projective cubics under projective coordinates. Then, we part from singular curves and focus on plane curves, studying their topological structure with the degree-genus theorem and their relation with Riemann surfaces with Riemann-Roch's theorem.

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Chapter 1

Introduction

The set of zeros of a polynomial allows us to understand relations between algebra, topology, and complex analysis. In particular, we call the set of zeros of a polynomial over an algebraic field an algebraic curve. Algebraic curves are related to Riemann surfaces and, thus, they relate the polynomial that defines the curve (algebra) with the orientable surface associated to the Riemann surface (topology) and with the functions and differentials defined on the Riemann surface (complex analysis).

We will study algebraic curves not only because they are one of the most beautiful areas of mathematics, but because a mathematic undergraduate already possesses most of the mathematical machinery required to study them. For instance, consider that we can represent a polynomial in $\mathbb{C}$ by multiplying the linear factors of each of its zeros. The zeros of the polynomial $P(x) = x^3 - 2x^2 - 5x + 6$ in $\mathbb{C}$, for example, are the points -2, 1, and 3; their linear factors are $(x-1)$, $(x+2)$, and $(x-3)$; and their product is $(x-1)(x+2)(x-3) = P(x)$. In fields with more dimensions than $\mathbb{C}$, however, polynomials have an infinite number of zeros. The zeros of the polynomial $P(x, y) = x - 2y$ in $\mathbb{C}^2$, for example, are the points $(2, 1)$, $(4, 2)$, $(6, 3)$, ... They are scalar multiples of each other and, therefore, they represent a line. The points in an algebraic curve can live in many different fields but, in this end-of-degree project, we are going to focus on $\mathbb{C}^2$ (affine curves) and on $\mathbb{P}^2$ (projective curves). The projective plane $\mathbb{P}^2$ is the set of lines in $\mathbb{C}^3$ that contain the point $(0, 0)$ and, therefore, it is the perfect field to represent the zeros of polynomials in multiple variables.

We are going to pay special attention to projective curves without singularities (plane curves). A singularity is a point in a curve that nullifies the derivatives of the polynomial that defines the curve. The curve defined by the polynomial $P(x, y) = (x - y)^2$, for example, has a singularity at the point $(1, 1)$. Plane curves can be thought of as well-behaved curves.

We will: define basic properties of affine and projective curves, study the intersection between two projective curves, and study the topological structure and the Riemann surfaces associated to plane curves.

Chapter 2

Curves

2.1 Affine curves

We start by introducing algebraic curves over $\mathbb{C}^2$.

Definition 2.1.1. An affine curve in $\mathbb{C}^2$ is the set of zeros of a non-constant polynomial in two variables with complex coefficients and without repeated factors.

We include the requirement that the polynomials cannot have repeated factors because of Hilbert's Nullstellestaz, a crucial result that describes when two polynomials define the same affine curve.

Theorem 2.1.2 (Hilbert's Nullstellestaz)**.** Two complex polynomials $P(x, y)$ and $Q(x, y)$ define the same affine curve if and only if they have the same irreducible factors up to multiplicity; that is, if and only if there exist positive integers m and n such that $P(x, y)$ divides $Q(x, y)^m$ and $Q(x, y)$ divides $P(x, y)^n$.

The situaton is simpler when the polynomials do not have repeated factors.

Corollary 2.1.2.1. Two complex polynomials without repeated factors define the same affine curve in $\mathbb{C}^2$ if and only if they are (nonzero) scalar multiples of each other.

At times, however, restricting affine curves to not have repeated factors is too limiting, and we will have to use the following, more general remark.

Remark 2.1.3. A complex algebraic curve is the class of equivalence of non-constant complex polynomials in which two polynomials are equivalent if and only if they are scalar multiples of each other. A polynomial with repeated factors defines the same curve but with multiplicities attached.

Example 2.1.4. Let $P(x, y) = y - x^2$, $Q(x, y) = 3(y - x^2)$, and $R(x, y) = (y - x^2)^3$. The polynomials $P(x, y)$ and $Q(x, y)$ define the exact same affine curve. The polynomial $R(x, y)$ defines the same affine curve as $P(x, y)$ and $Q(x, y)$ but with multiplicity 3.

Next follow a few definitions that will come in handy. For them, let $\tilde{C}$ be an affine curve defined by a polynomial $P(x, y)$.

Definition 2.1.5. The degree of $\tilde{C}$ is the degree of the polynomial $P(x, y)$.

The degree of an affine curve $\tilde{C}$ does not depend on the representative in the class of equivalence of polynomials that define $\tilde{C}$. The polynomials $P(x, y) = y - x^2$ and $Q(x, y) = 3(y - x^2)$, for example, both represent the same affine curve of degree $\deg P = \deg Q = 2$.

Definition 2.1.6. A point $(a, b) \in \tilde{C}$ is singular if and only if $\frac{\partial P}{\partial x}(a, b) = 0 = \frac{\partial P}{\partial y}(a, b)$. The set of singular points of $\tilde{C}$ is denoted $Sing(\tilde{C})$ and $\tilde{C}$ is nonsingular if it does not have any singular points.

Definition 2.1.7. The multiplicity of $\tilde{C}$ at a point $(a, b) \in \tilde{C}$ is the number of times $\tilde{C}$ passes through (a, b); mathematically, it is the smallest possible integer m such that

$$\frac{\partial^m P}{\partial x^i \partial y^j}(a, b) \neq 0$$

for some $i, j \geq 0$ where $i + j = m$.

Definition 2.1.8. The irreducible factors of $P(x, y)$ define the irreducible factors of $\tilde{C}$. The affine curve $\tilde{C}$ is then irreducible if $P(x, y)$ is irreducible.

2.1.1 Examples

We introduce now three curves that we will follow throughout the chapter. They are the cusp, node, and club, and we will use them to illustrate the different concepts we introduce. Figures 2.1, 2.2, and 2.3 represent their real parts so that we can get a rough idea of what they look like. We cannot represent them fully because they live in the four-real-dimensional vector space $\mathbb{C}^2$.

Cusp

In $\mathbb{C}^2$ the cusp is the affine curve defined by the polynomial $P(x, y) = x^3 - y^2$ with partial derivatives $\frac{\partial P}{\partial x}(x, y) = 3x^2$ and $\frac{\partial P}{\partial y}(x, y) = -2y$. Its degree is 3 and it only has one singularity: ordinary of multiplicity 1 at the point $(0, 0)$.

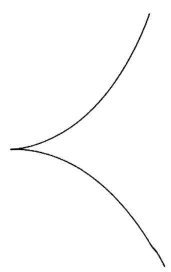

Figure 2.1: Representation of the cusp in $\mathbb{R}^2$.

Node

Similarly, in $\mathbb{C}^2$ the node is the affine curve defined by the polynomial $Q(x,y) = x^3 + x^2 - y^2$ with partial derivatives $\frac{\partial Q}{\partial x}(x,y) = 3x^2 + 2x$ and $\frac{\partial Q}{\partial y}(x,y) = -2y$. Its degree is also 3 and it only has one singularity: ordinary of multiplicity 2 at the point $(0,0)$.

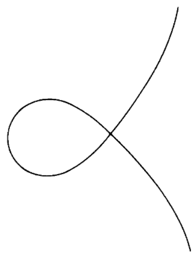

Figure 2.2: Representation of the node in $\mathbb{R}^2$.

Club

Finally, in $\mathbb{C}^2$ the club is the affine curve defined by the polynomial $R(x,y) = (x^4+y^4)^2 - x^2y^2$ with partial derivatives $\frac{\partial R}{\partial x}(x,y) = 8x^7 + 8x^3y^4 - 2xy^2$ and $\frac{\partial R}{\partial y}(x,y) = 8y^7 + 8x^4y^3 - 2x^2y$. Its degree is 8 and it only has one singularity: ordinary of multiplicity 2 at the point $(0,0)$.

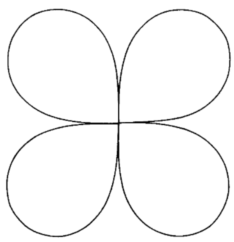

Figure 2.3: Representation of the club in $\mathbb{R}^2$.

2.2 The projective space

Up until now we have been working in $\mathbb{C}^2$. The algebraic field $\mathbb{C}^2$ can be very useful but it is not compact, which can be problematic. Consider the following example: the affine curves $y^2 = x^2 - 1$ and $y = cx$, $c \in \mathbb{C}$, intersect at two points when $c \neq \pm 1$, but they are asymptopic when $c = \pm 1$. We could make them always intersect at two points by saying that when $c = \pm 1$ they intersect at infinity. Thus, we are going to projectivize affine curves in $\mathbb{C}^2$ by adding points at infinity.

We will represent a point $(x,y) \in \mathbb{C}^2$ with the line in $\mathbb{C}^3$ that passes through the points $(0,0,0)$ and $(x,y,1)$. As (x,y) tends to infinity, the scope of the line that represents it tends to 0, so it makes sense to represent points at infinity with the lines contained in $z = 0$ (i.e. the lines that pass through the points $(0,0,0)$ and $(x,y,0)$).

We define the projective space as a set of equivalence classes

$$\mathbb{P}^n_{\mathbb{C}} = \{[x_0 : ... : x_n] = (x_0, ..., x_n) \in \mathbb{C}^{n+1} - \{0\}\}$$

in which two points are equivalent if and only if they are contained in the same line:

$$[x_0 : ... : x_n] = [y_0 : ... : y_n] \iff \exists \lambda \in \mathbb{C} - \{0\} : x_i = \lambda y_i, \forall i \in \{0, ..., n\}.$$

In accordance with the intuition we discussed above, $\mathbb{P}^n_{\mathbb{C}}$ consists of two different subsets of lines: the lines $U_n = \{[x_0 : ... : x_n] \in \mathbb{P}^n_{\mathbb{C}} : x_n = 1\}$ that intersect the plane $x_n = 1$ and represent $\mathbb{C}^n$, and the lines $\bar{U}_n = \{[x_0 : ... : x_n] \in \mathbb{P}^n_{\mathbb{C}} : x_n = 0\}$ that are contained in $x_n = 0$ and represent the points at infinity.

To make sure that $\mathbb{P}^n_{\mathbb{C}}$ is compact, we endow it with a topology. In particular, we use the natural projection Π to grant $\mathbb{P}^n_{\mathbb{C}}$ the quotient topology induced from the usual complex analytic topology in $\mathbb{C}^{n+1} - \{0\}$.

Remark 2.2.1. The natural projection $\Pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n_{\mathbb{C}}$ relates points in $\mathbb{C}^{n+1} - \{0\}$ with lines in $\mathbb{P}^n_{\mathbb{C}}$: $\Pi(x_0, ..., x_n) = [x_0 : ... : x_n]$, and has two important properties:

- A subset A of $\mathbb{P}^n_{\mathbb{C}}$ is open (or closed) if and only if $\Pi^{-1}(A)$ is an open (or closed) subset of $\mathbb{C}^{n+1} - \{0\}$.
- Π is continuous.

Theorem 2.2.2. The projective space $\mathbb{P}^n_{\mathbb{C}}$ is compact and Hausdorff.

Proof. The proof can be found in propositions 2.18 and 2.23 of [1]. □

2.3 Projective curves

We now introduce complex projective curves. The main difference between complex affine curves and complex projective curves is that complex affine curves are subsets of $\mathbb{C}^2$ while complex projective curves are subsets of $\mathbb{P}^2_{\mathbb{C}}$.

As we saw in the previous section, the elements of $\mathbb{P}^2_{\mathbb{C}}$ are lines, so we can no longer define complex projective curves with any polynomial. We have to use homogeneous polynomials. What makes homogeneous polynomials the perfect candidate to define complex projective curves is the fact that the zero set they define is invariant by rescaling, $P(\lambda x, \lambda y, \lambda z) = 0 \iff P(x, y, z) = 0$ for every $\lambda \in \mathbb{C} - \{0\}$, so it is easy to think of their input space as a set of lines.

Definition 2.3.1. A polynomial $P(x_1, ..., x_n)$ of degree d is homogeneous if

$$P(\lambda x_1, ..., \lambda x_n) = \lambda^d P(x_1, ..., x_n)$$

for every $\lambda \in \mathbb{C}$.

An advantage of homogeneous polynomials is that they can be decomposed into linear factors.

Lemma 2.3.2. A homogeneous polynomial $P(x, y)$ of degree d can be decomposed as the product of d linear factors $(\alpha_i x + \beta_i y)$ for some α_i, $\beta_i \in \mathbb{C}$.

Proof. Let $P(x, y)$ be a homogeneous polynomial. There must exist complex numbers α_0, ..., α_d not all zero such that

$$P(x, y) = \prod_{r=0}^{d} \alpha_r x^r y^{d-r} = y^d \prod_{r=0}^{d} \alpha_r \left(\frac{x}{y}\right)^r. \tag{2.1}$$

If e be the biggest integer in $\{0, ..., d\}$ such that $\alpha_e \neq 0$, then $\prod_{r=0}^{d} \alpha_r (\frac{x}{y})^r$ is a polynomial of degree e in one variable, $\frac{x}{y}$, so by the Fundamental Theorem of Algebra we can find complex numbers γ_0, ..., γ_e to decompose $\prod_{r=0}^{d} \alpha_r (\frac{x}{y})^r$ as the product of e linear factors:

$$\prod_{r=0}^{d} \alpha_r \left(\frac{x}{y}\right)^r = \alpha_e \prod_{i=1}^{e} \left(\frac{x}{y} - \gamma_i\right). \tag{2.2}$$

We can combine equations (2.1) and (2.2) to get

$$P(x, y) = \alpha_e y^d \prod_{i=1}^{e} \left(\frac{x}{y} - \gamma_i\right) = \alpha_e \, y^{d-e} \prod_{i=1}^{e} (x - \gamma_i y),$$

a decomposition of $P(x, y)$ as the product of d linear factors. □

After having introduced homogeneous polynomials, we are ready to define complex projective curves.

Definition 2.3.3. A complex projective curve is the zeros in $\mathbb{P}^2_{\mathbb{C}}$ of a nonconstant, homogeneous, complex polynomial without repeated factors.

The properties of complex projective curves are very similar to those of complex algebraic curves.

Definition 2.3.4. Let C be a complex projective curve defined by a homogeneous polynomial $P(x, y, z)$.

- The degree of C is the degree of $P(x, y, z)$.
- The irreducible factors of C are the projective curves defined by an irreducible polynomial that divides $P(x, y, z)$. The curve C is then irreducible if $P(x, y, z)$ is irreducible.
- A point $[a : b : c] \in C$ is singular if and only if

$$\frac{\partial P}{\partial x}(a, b, c) = \frac{\partial P}{\partial y}(a, b, c) = \frac{\partial P}{\partial z}(a, b, c) = 0.$$

Furthermore, as in the affine case, two homogeneous polynomials without repeated factors define the same complex projective curve if and only if they are scalar multiples of each other, and a homogeneous polynomial with repeated factors can be thought of as defining a projective curve with multiplicities attached.

Example 2.3.5. Let $P(x,y,z) = yz - x^2$, $Q(x,y,z) = 3(yz - x^2)$, and $R(x,y,z) = (yz-x^2)^3$. The polynomials $P(x,y,z)$ and $Q(x,y,z)$ define the exact same projective curve. The polynomial $R(x,y,z)$ defines the same projective curve as $P(x,y,z)$ and $Q(x,y,z)$ but with multiplicity 3.

Definition 2.3.6. Let α, β, γ be complex numbers not all zero. A projective line is a projective curve defined by the equation $\alpha x + \beta y + \gamma z = 0$.

Lemma 2.3.7. A complex projective curve in $\mathbb{P}^2_{\mathbb{C}}$ is compact and Hausdorff.

Proof. Let C be a complex projective curve in $\mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous polynomial $P(x,y,z)$. The curve C is Hausdorff because $\mathbb{P}^2_{\mathbb{C}}$ is Hausdorff, which is a hereditary property, and since $\mathbb{P}^2_{\mathbb{C}}$ is compact, we can show that C is compact by showing that C is closed. We know that $\Pi^{-1}(C) = \{(x,y,z) \in \mathbb{C}^3 - \{0\} : P(x,y,z) = 0\}$ is closed since it is the inverse image of a closed subset ($\{0\}$) by a continuous function ($P(x,y,z)$), and, therefore, by remark 2.2.1, C must be closed. $\square$

A complex affine curve is a complex projective curve without ‘points at infinity’. We can get the associated complex affine curve of a complex projective curve by ignoring the lines that are contained in the plane $z = 0$ by, for example, setting $z = 1$. Moreover, if we have a complex affine curve defined by a polynomial $P(x,y)$ of degree d, we can get its associated complex projective curve by projectivizing $P(x,y)$ (that is: by multiplying the terms of $P(x,y)$ with degree $r < d$ by z^{d-r}).

2.3.1 Examples

The curves we introduced in the previous example were affine. We can make them projective by making their polynomials homogeneous and, with it, adding ‘points at infinity’.

Cusp

The cusp in $\mathbb{C}^2$ is the affine curve defined by the polynomial $P(x,y) = x^3 - y^2$. We can projectivize the cusp by making $P(x,y)$ homogeneous: $P(x,y,z) = x^3 - y^2 z$, and we can get the affine version back by setting z to 1: $P(x,y,1) = x^3 - y^2$.

Node

Similarly, the node in $\mathbb{C}^2$ is the affine curve defined by the polynomial $Q(x,y) = x^3 + x^2 - y^2$. We can projectivize the node by making $Q(x,y)$ homogeneous: $Q(x,y,z) = x^3 + x^2 z - y^2 z$, and we can get the affine version back by setting z to 1: $Q(x,y,1) = x^3 + x^2 - y^2$.

Club

Finally, the club in $\mathbb{C}^2$ is the affine curve defined by the polynomial $R(x,y) = (x^4+y^4)^2 - x^2y^2$. We can projectivize the club by making $R(x,y)$ homogeneous: $R(x,y,z) = (x^4+y^4)^2 - x^2y^2z^4$, and we can get the affine version back by setting z to 1: $R(x,y,1) = (x^4+y^4)^2 - x^2y^2$.

These examples do not have any images because complex projective curves are too hard to visualize. As we are now working in $\mathbb{P}^2_{\mathbb{C}}$, we would have to represent its six dimensions in a two-dimensional image, which would not be helpful.

We end the section by stating two important relations.

Lemma 2.3.8 (Euler's relation)**.** If $P(x,y,z)$ is a homogeneous polynomial of degree m, then it can be expressed in terms of its partial derivatives:

$$x\frac{\partial P}{\partial x}(x,y,z) + y\frac{\partial P}{\partial y}(x,y,z) + z\frac{\partial P}{\partial z}(x,y,z) = mP(x,y,z).$$

Proof. Let $P(x,y,z)$ be a homogeneous polynomial of degree m. We get Euler's relation by differentiating the equation $P(\lambda x, \lambda y, \lambda z) = \lambda^m P(x,y,z)$ with respect to λ and then settiñg λ to 1. □

As we have seen, complex affine curves are a subset of complex projective curves. They are, however, a special subset: in them we can read the singularities of complex projective curves. We can formalize this relation as follows:

Lemma 2.3.9. Let C be a projective curve defined by a polynomial $P(x,y,z)$. A point $[a:b:c] \in C$ with $c \neq 0$ is a singularity of C if and only if the point $(\frac{a}{c}, \frac{b}{c})$ is a singularity of the affine curve defined by $P(x,y,1)$.

Proof. Let C be a projective curve defined by a homogeneous polynomial $P(x,y,z)$ and let $\tilde{C}$ be the affine curve defined by $P(x,y,1)$. The point $(\frac{a}{c}, \frac{b}{c})$ is a singularity of $\tilde{C}$ if and only if

$$P(\frac{a}{c}, \frac{b}{c}, 1) = 0 = \frac{\partial P}{\partial x}(\frac{a}{c}, \frac{b}{c}, 1) = \frac{\partial P}{\partial y}(\frac{a}{c}, \frac{b}{c}, 1). \tag{2.3}$$

The polynomial $P(x,y,z)$ and its partial derivatives are homogeneous, so we can rescale the input by c in equation (2.3) to get

$$P(a,b,c) = 0 = \frac{\partial P}{\partial x}(a,b,c) = \frac{\partial P}{\partial y}(a,b,c). \tag{2.4}$$

We can now combine equation (2.4) with Euler's relation (lemma 2.3.8) to get that $\frac{\partial P}{\partial z}(a,b,c) = 0$, and, therefore, that the point $[a:b:c]$ is a singularity of C. □

Chapter 3

Algebra

3.1 Intersections between projective curves

To get a deeper understanding of complex projective curves we are going to study how they intersect. In particular, we want to be able to know how to count the number of points of intersection between two projective curves of arbitrary degree. We can get some intuition by counting the number of points of intersection between two curves of low degree.

Example 3.1.1. Let C_1 and D_1 be the lines defined by the polynomials $P_1(x, y, z) = 2x + y + z$ and $Q_1(x, y, z) = x + y + z$; and let C_2 and D_2 be the conics defined by the polynomials $P_2(x, y, z) = x^2 + yz$ and $Q_2(x, y, z) = x^2 + (y + z)^2$.

The lines C_1 and D_1 intersect at one point with multiplicity one: $[0 : 1 : -1]$. The line C_1 and the conic C_2 intersect at two points with multiplicity one: $[1 - \sqrt{2} : 2\sqrt{2} - 3 : 1]$ and $[1 + \sqrt{2} : -2\sqrt{2} - 3 : 1]$. The line D_1 is tangent to the conic D_2, so they intersect at one point with multiplicity two: $[0 : 1 : -1]$. Finally, the conics C_2 and D_2 intersect at two points with multiplicity two: $[\sqrt{2 + 2i\sqrt{3}} : -1 - i\sqrt{3} : 2]$ and $[\sqrt{2 - 2i\sqrt{3}} : i\sqrt{3} - 1 : 2]$.

As we have just seen, we can find the points of intersection between two curves of low degree easily. For curves of higher degree, however, we need to come up with a more robust method: the resultant.

The resultant of two polynomials is another polynomial that vanishes at their common roots. The resultant projects both polynomials over their first irreducible factor and, with it, compresses the intersection points into the zeros of a polynomial expression. Here is a more precise definition:

Definition 3.1.2. The resultant of two polynomials $P(x) = a_0 + a_1x + ... + a_nx^n$ and $Q(x) = b_0 + b_1x + ... + b_mx^m$ with complex coefficients such that $a_n, b_m \neq 0$ is

$$\mathcal{R}_{P,Q} = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_n & 0 & 0 & & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & 0 & & \cdots & 0 \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ 0 & 0 & \cdots & 0 & a_0 & a_1 & & \cdots & a_n \\ b_0 & b_1 & & \cdots & & b_m & 0 & 0 \cdots & 0 \\ 0 & b_0 & b_1 & & \cdots & & b_m & 0 \cdots & 0 \\ & & & & \cdots & & & & \\ 0 & \cdots & 0 & b_0 & b_1 & & & \cdots & b_m \end{pmatrix}.$$

To understand how the resultant works, consider the following example.

Example 3.1.3. Let C and D be the projective curves defined by the polynomials $P(x, y, z) = xy - z^2$ and $Q(x, y, z) = x^2 + y^2 - z^2$. To find their intersection points we treat them as polynomials in x with coefficients in the ring $\mathbb{C}[y, z]$ and we calculate their resultant:

$$\det \begin{pmatrix} -z^2 & y & 0 \\ 0 & -z^2 & y \\ y^2 - z^2 & 0 & 1 \end{pmatrix} = z^4 + y^2(y^2 - z^2),$$

Note that, as we discussed in section 2.2, z can either be 0 or 1. If $z = 0$, then x and y would both have to be 0 and, since $[0 : 0 : 0] \notin \mathbb{P}^2_{\mathbb{C}}$, there would be no intersection points. Thus, let $z = 1$. The roots of $\mathcal{R}_{P,Q}(y, 1) = y^4 - y^2 + 1$ are

$$y = \pm\sqrt{\frac{1 \pm 3i}{2}},$$

and if we plug them into one of the polynomials, then we get the intersection points between C and D:

$$\left(\pm\sqrt{\frac{2}{1 \pm 3i}}, \pm\sqrt{\frac{1 \pm 3i}{2}} \right).$$

All the intersection points between C and D have complex coordinates, which explains why when we represent the curves in $\mathbb{R}^2$ (as in Figure 3.1) they do not intersect.

The following two results illustrate why the resultant is so important.

Lemma 3.1.4. Two single variable polynomials have a nonconstant common factor if and only if their resultant is zero.

Proof. Two polynomials $P(x) = a_0 + a_1x + ... + a_nx^n$ and $Q(x) = b_0 + b_1x + ... + b_mx^m$ of degrees n and m only have a common factor $R(x)$ if there exist two nonzero polynomials $\phi(x) = \alpha_0 + \alpha_1x + ... + \alpha_{n-1}x^{n-1}$ and $\psi(x) = \beta_0 + \beta_1x + ... + \beta_{m-1}x^{m-1}$ with complex coefficients such that $P(x) = R(x)\phi(x)$ and $Q(x) = R(x)\psi(x)$. The

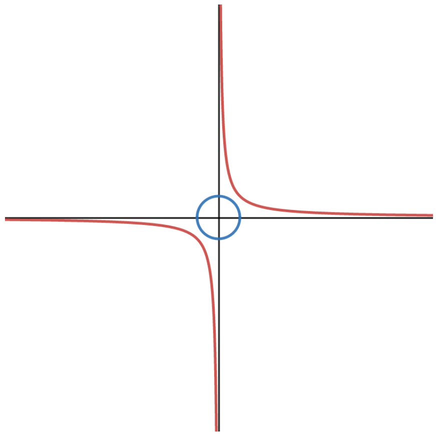

Figure 3.1: The real parts of C (in red) and D (in blue) when $z = 1$.

common factor $R(x)$ exists if and only if $P(x)\psi(x) = Q(x)\phi(x)$, or, equivalently, we can equate the coefficients of x to get

$$\begin{aligned} a_0\beta_0 &= b_0\alpha_0 \\ a_0\beta_1 + a_1\beta_0 &= b_0\alpha_1 + b_1\alpha_0 \\ &\vdots \\ a_n\beta_{m-1} &= b_m\alpha_{n-1}. \end{aligned}$$

A nonzero solution $(\alpha_0, ..., \alpha_{n-1}, \beta_0, ..., \beta_{m-1})$ to this system of equations exists if and only if the determinant that defines $\mathcal{R}_{P,Q}$ is zero. □

Remark 3.1.5. Two multivariable homogeneous polynomials also have a non-constant common factor if and only if their resultant is zero, as long as their degree does not change when regarded as polynomials in x with coefficients in the ring $\mathbb{C}[y, z]$. That is, as long as the point $[1 : 0 : 0]$ is not a zero of either of them.

Lemma 3.1.6. The resultant of two homogeneous polynomials of degrees n and m is a homogeneous polynomial of degree nm.

Proof. The proof can be found in [1] as the proof of lemma 3.7. □

Thanks to these results we can bound the number of points of intersection between two projective curves.

Theorem 3.1.7. Two projective curves intersect in at least one point.

Proof. Let C and D be two projective curves defined by the homogeneous polynomials $P(x, y, z)$ and $Q(x, y, z)$ of degrees n and m. By lemma 3.1.6 their resultant $\mathcal{R}_{P,Q}(y, z)$ is a homogeneous polynomial of degree nm and, by lemma 2.3.2, it is either zero or it can be decomposed as the product of nm linear factors of the form $cy - bz$, with $b, c \in \mathbb{C} - \{0\}$. In either case there is at least one point (b, c) such that $\mathcal{R}_{P,Q}(b, c) = 0$, which by lemma 3.1.4 implies that C and D have a common factor, $P(a, b, c) = 0 = Q(a, b, c)$. □

The lower bound, as insignificant as it may seem, uncovers a relation between singularities and reducibility. Namely:

Corollary 3.1.7.1. Every non-singular projective curve is irreducible.

Proof. To prove corollary 3.1.7.1, we are going to show that every reducible projective curve has at least one singularity. Let C be a projective curve defined by a polynomial $P(x, y, z)$ that can be expressed as the product of two homogeneous polynomials, $R(x, y, z)$ and $S(x, y, z)$. By theorem 3.1.7 there exists at least one point $[a : b : c] \in C$ such that $R(a, b, c) = 0 = S(a, b, c)$. The partial derivative of $P(x, y, z)$ with respect to x,

$$\frac{\partial P}{\partial x}(x, y, z) = \frac{\partial R}{\partial x}(x, y, z)S(x, y, z) + R(x, y, z)\frac{\partial S}{\partial x}(x, y, z),$$

then vanishes at $[a : b : c]$ and, similarly, the partial derivatives with respect to y and z also vanish at $[a : b : c]$. The point $[a : b : c]$ is therefore a singularity of C. □

We now give the upper bound for the number of points of intersection between two projective curves.

Theorem 3.1.8 (Weak form of Bézout's theorem)**.** Two projective curves of degrees n and m without common irreducible factors intersect in at most nm points.

Proof. Let C and D be two complex projective curves defined by the polynomials $P(x, y, z)$ and $Q(x, y, z)$ of degrees n and m. We are going to prove that if C and D intersect in more than nm points, then they have a common irreducible factor.

Let S be a set of $nm + 1$ distinct points of intersection between C and D. We start by making a projective transformation such that the point $[1 : 0 : 0]$ is not in C, D, or in any of the lines that contain two points of S. By lemma 3.1.4 the resultant between $P(x, b, c)$ and $Q(x, b, c)$ is zero at every point $[a : b : c] \in S$, so $\mathcal{R}_{P,Q}(y, z)$ must have $nm + 1$ linear factors of the form $cy - bz$; no two of these linear factors are scalar multiples of each other since that would imply that they both lie on the same line as $[1 : 0 : 0]$ – a contradiction. But $\mathcal{R}_{P,Q}(y, z)$ cannot have $nm + 1$ distinct linear factors since by lemma 3.1.6 it is a homogeneous polynomial of degree nm, and by lemma 2.3.2 it is then either zero or the product of nm linear factors. Thus, $\mathcal{R}_{P,Q}(y, z)$ must be zero and, with it, by remark 3.1.5 C and D must have a common irreducible factor. □

The upper bound uncovers yet another relation between singularities and reducibility.

Corollary 3.1.8.1. An irreducible projective curve has at most finitely many singularities.

Proof. Let C be an irreducible projective curve defined by a polynomial $P(x, y, z)$ and let D be the curve defined by the polynomial $\frac{\partial P}{\partial x}(x, y, z)$. Every singularity of C lies in $C \cap D$ and, since both C and D are irreducible, $C \cap D$ contains a finite number of points by theorem 3.1.8. □

The bounds for the number of intersection points give us a lot of insight into how two curves intersect. They also appear to suggest that there may be a formula that precisely determines the number of intersection points between two curves. To see if this is in fact the case, we introduce a second tool: the intersection multiplicity.

The intersection multiplicity formalizes the idea that two curves can intersect at the same point several times. Before we introduce it, however, we define a coordinate system that will come in handy later on.

Definition 3.1.9. Let C and D be two projective curves. We define a coordinate system such that:

- $[1 : 0 : 0] \notin C \cup D$,
- $[1 : 0 : 0]$ does not lie on any line containing two distinct points of $C \cap D$, and
- $[1 : 0 : 0]$ does not lie on the tangent line to C or D at any point of $C \cap D$.

Here is an example that illustrates the existence of this coordinate system.

Example 3.1.10. Let C and D be two projective curves defined by the polynomials $P(x, y, z) = (y + z)(y + z - x)$ and $Q(x, y, z) = y + 2z$. The curves C and D intersect at the points $[1 : 0 : 0]$ and $[1 : 2 : -1]$. After the projective transformation $[x : y : z] \to [y + z : x : z]$, however, $P(x, y, z) = x(x - y)$, $Q(x, y, z) = x + z$, and the intersection points between C and D become $[0 : 1 : 0]$ and $[1 : 1 : -1]$.

We are now in the coordinate system defined in definition 3.1.9, since:

- $[1 : 0 : 0] \notin C \cup D$;
- $[1 : 0 : 0] \notin D$, the only line that contains two distinct points of $C \cap D$; and
- $[1 : 0 : 0]$ is not in the tangent line to C at $[0 : 1 : 0]$, $x = 0$, or at $[1 : 1 : -1]$, $x = y$.

Definition 3.1.11. Let C and D be two projective curves defined by the homogeneous polynomials $P(x, y, z)$ and $Q(x, y, z)$. The intersection multiplicity at a point $p \in \mathbb{P}^2_{\mathbb{C}}$ is a function $\mathbf{I}_p : (C, D) \mapsto \mathbb{N} \cup \{0\}$ such that:

1. $\mathbf{I}_p(C, D) = 0 \iff p \notin C \cap D$;
2. $\mathbf{I}_p(C, D) = \infty \iff p$ lies in a common irreducible factor of C and D;
3. if C and D are distinct lines and $p \in C \cap D$, then $\mathbf{I}_p(C, D) = 1$;
4. $\mathbf{I}_p(C, D) = \mathbf{I}_p(D, C)$;
5. if C_1 and C_2 are projective curves, then $\mathbf{I}_p(C, D) = \mathbf{I}_p(C_1, D) + \mathbf{I}_p(C_2, D) \iff C = C_1 \cup C_2$;

and such that if $R(x, y, z)$ is a homogeneous polynomial of degree $m - n$ and we abuse the notation by saying that $\mathbf{I}_p(P, Q)$ is equivalent to $\mathbf{I}_p(C, D)$, then

6. $\mathbf{I}_p(P, PR + Q) = \mathbf{I}_p(P, Q)$.

Moreover, if we are under the projective coordinates defined in definition 3.1.9 and C and D do not have a common irreducible factor, then the intersection multiplicity $\mathbf{I}_p(C, D)$ at every point $p = [a, b, c] \in C \cap D$ is the largest integer k such that $(bz - cy)^k$ divides the resultant $\mathcal{R}_{P,Q}(y, z)$.

Theorem 3.1.12. The intersection multiplicity between two projective curves exists and is unique at every point in $\mathbb{P}^2_{\mathbb{C}}$.

Proof. Here we go through the algorithm to calculate the intersection multiplicity between two projective curves. A rigurous proof of theorem 3.1.12 is given by [1] as the proof of theorem 3.18.

Let C and D be two projective curves and let $p \in \mathbb{P}^2_{\mathbb{C}}$. To calculate $\mathbf{I}_p(C, D)$ we apply an algorithm based on two steps. First we:

1. apply (5);
2. apply a combination of (4) and (6) to each of the intersection multiplicities that result from step 1; and
3. take the intersection multiplicities that, after step 2, contain a reducible curve back to step 1.

The intersection multiplicity $\mathbf{I}_p(C, D)$ will now be expressed as a sum of intersection multiplicities between lines without common irreducible factors. At each point $p \in \mathbb{P}^2_{\mathbb{C}}$, the intersection multiplicity between two lines without common irreducible factors is then one if p is an intersection point of the lines and zero otherwise. □

Remark 3.1.13. Let C and D be two projective curves and let p be a point in $\mathbb{P}^2_{\mathbb{C}}$. The intersection multiplicity $\mathbf{I}_p(C, D)$ depends only on the irreducible factors of C and D that contain p.

Example 3.1.14. Let C and D be two projective curves defined by the polynomials $P(x, y, z) = z^2(x^2 - y^2)$ and $Q(x, y, z) = x^3 - y^2 z$. As an example we calculate the intersection multiplicity $\mathbf{I}_p(C, D) = \mathbf{I}_p(z^2(x^2 - y^2), x^3 - y^2 z)$ at each point $p \in C \cap D$.

First we apply (5) three times:

$$\mathbf{I}_p(z^2(x^2 - y^2), x^3 - y^2 z) = 2 \cdot \mathbf{I}_p(z, x^3 - y^2 z) + \mathbf{I}_p(x - y, x^3 - y^2 z) + \mathbf{I}_p(x + y, x^3 - y^2 z).$$

Then, we replace $x^3 - y^2 z$ with $(x - y)(x^2 + yz) + xy(x - z)$ in the second term and with $(x + y)(x^2 - yz) + xy(z - x)$ in the third term,

$$\mathbf{I}_p(x-y, x^3-y^2z) = \mathbf{I}_p(x-y, (x-y)(x^2+yz)+xy(x-z))$$
$$\mathbf{I}_p(x+y, x^3-y^2z) = \mathbf{I}_p(x+y, (x+y)(x^2-yz)+xy(z-x)),$$

and we apply (6) to all three terms,

$$\mathbf{I}_p(z, x^3-y^2z) = \mathbf{I}_p(z, x^3)$$
$$\mathbf{I}_p(x-y, (x-y)(x^2+yz)+xy(x-z)) = \mathbf{I}_p(x-y, xy(x-z))$$
$$\mathbf{I}_p(x+y, (x+y)(x^2-yz)+xy(z-x)) = \mathbf{I}_p(x+y, xy(z-x))$$

Finally, we apply (5) three times to every term:

$$\mathbf{I}_p(z, x^3) = 3\cdot\mathbf{I}_p(z, x)$$
$$\mathbf{I}_p(x-y, xy(x-z)) = \mathbf{I}_p(x-y, x)+\mathbf{I}_p(x-y, y)+\mathbf{I}_p(x-y, x-z)$$
$$\mathbf{I}_p(x+y, xy(z-x)) = \mathbf{I}_p(x+y, x)+\mathbf{I}_p(x+y, y)+\mathbf{I}_p(x+y, z-x)$$

The expression for $\mathbf{I}_p(C, D)$ is then

$$\begin{aligned}\mathbf{I}_p(C, D) = 6\cdot\mathbf{I}_p(z, x) &+ \mathbf{I}_p(x-y, x)+\mathbf{I}_p(x-y, y)+\mathbf{I}_p(x-y, x-z)\\ &+ \mathbf{I}_p(x+y, x)+\mathbf{I}_p(x+y, y)+\mathbf{I}_p(x+y, z-x),\end{aligned}$$

the sum of the intersection multiplicities between different lines. We can calculate $\mathbf{I}_p(C, D)$ at each point $p \in C \cap D$ with the expression by seeing what lines intersect at p. Thus, if $p = [0 : 1 : 0]$, then $\mathbf{I}_p(C, D) = 6$; if $p = [0 : 0 : 1]$, then $\mathbf{I}_p(C, D) = 4$; if $p = [1 : 1 : 1]$, then $\mathbf{I}_p(C, D) = 1$; and if $p = [1 : -1 : 1]$, then $\mathbf{I}_p(C, D) = 1$.

We can now ask how many times two projective curves intersect and, as we had hoped, the answer is given by a precise formula.

Theorem 3.1.15 (Bézout's theorem)**.** The sum of the intersection multiplicities of two projective curves C and D of degrees n and m without common irreducible factors is

$$\sum_{p\in C\cap D} \mathbf{I}_p(C, D) = nm.$$

Proof. Let C and D be two complex projective curves defined by the homogeneous polynomials $P(x, y, z)$ and $Q(x, y, z)$ of degrees n and m, and, via a projective transformation, let us assume that our coordinate system is like the one defined in definition 3.1.9.

By remark 3.1.4 we know that the resultant of C and D is not zero because C and D do not have common irreducible factors; furthermore, by lemma 3.1.6 their resultant is a homogeneous polynomial of degree nm. By lemma 2.3.2 we know that, if e_1, ..., e_k are positive integers such that $e_1 + ... + e_k = nm$, then $\mathcal{R}_{P,Q}(y,z)$ can be expressed as the product of nm linear factors

$$\mathcal{R}_{P,Q}(y,z) = \prod_{i=0}^{k} (c_i y - b_i z)^{e_i}.$$

These linear factors are not scalar multiples of each other because of our choice of coordinate system. Finally, as in the proofs of theorems 3.1.7 and 3.1.8, we know that for each pair (b_i, c_i) there exists an $a_i \in \mathbb{C}$ such that $p_i = [a_i : b_i : c_i] \in C \cap D$, and by section (6) of the definition of the intersection multiplicity (3.1.11)

$$\mathbf{I}_{p_i}(C,D) = e_i.$$

□

The following corollary states under what conditions two projective curves reach the upper bound of the number of intersection points.

Corollary 3.1.15.1. Let C and D be two projective curves of degrees n and m. The intersection $C \cap D$ consists of exactly nm points if the points in $C \cap D$ are non-singular points of C and D and the tangent lines to C and to D at them are distinct.

Proof. The proof is the same as the proof of corollary 3.25 of [1]. □

Example 3.1.16. The projective curves C and D defined by the polynomials $P(x,y,z) = y + z$ and $Q(x,y,z) = xz + y^2$ intersect at the nonsingular points $[1:0:0]$ and $[1:1:-1]$. The curve C is a projective line so its tangent lines at every point are C itself. The tangent lines to D are $z = 0$ at $[1:0:0]$ and $x = 2y+z$ at $[1:1:-1]$. Thus, the tangent lines at the points of intersection between C and D are distinct and, since the degrees of C and D are 1 and 2, by corollary 3.1.15.1 C and D intersect at exactly two points.

3.2 Low-degree irreducible curves

Bézout's theorem answers the question of how many points of intersection there are between two curves. It is a building block in the study of algebraic curves, and has many interesting applications. Two interesting applications of Bézout's theorem and the tools developed to prove it are theorems 3.2.2 and 3.2.9. They state that there exist projective transformations that make all irreducible projective conics equivalent and all irreducible projective cubics equivalent.

The study of conic curves goes all the way back to Ancient Greece. The hyperbola, the parabola, and the ellipse are indispensable to the study of geometry and appear in many different fields of mathematics.

Definition 3.2.1. A projective conic is a curve of degree two in $\mathbb{P}^2_{\mathbb{C}}$.

We can use the relations we have established between singularity and reducibility to prove that:

Theorem 3.2.2. Every irreducible projective conic is, under a projective transformation, non-singular and equivalent to the projective conic $x^2 = yz$.

Proof. Let α_1, ..., α_6 be complex numbers and let C be the irreducible conic defined by the polynomial $P(x, y, z) = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2 + \alpha_4 xy + \alpha_5 xz + \alpha_6 yz$. By corollary 3.1.8.1 C has at most finitely many singularities, so there exists a $\lambda \in \mathbb{C}$ such that the projective transformation $[x : y : z] \to [x + \lambda y : y : z]$ takes the point $[0 : 1 : 0]$ to a nonsingular point of C ($\alpha_2 = 0$) and its tangent line to C the line $z = 0$ ($\alpha_4 = 0$). After renaming the coefficients we get $P(x, y, z) = ayz + bx^2 + cxz + dz^2$.

Note that since C is irreducible, a and b cannot be zero. The projective transformation $[x : y : z] \to [\sqrt{b}x, ay + cx + dz, -z]$ then takes C to the non-singular projective conic $x^2 = yz$. □

The study of projective cubics is not as straight forward as the study of projective conics; we first have to introduce the points of inflection and the Hessian of a curve, and only then projective cubics and their equivalence relation under projective coordinates. We start with the points of inflection.

A point of inflection of a function is a point that makes the second derivatives of the funcion vanish. To translate this concept to curves we have to introduce the Hessian; with it we can say that a point of inflection of a curve is a point that makes the Hessian of the curve vanish.

Definition 3.2.3. The Hessian of a homogeneous polynomial $P(x, y, z)$ of degree d is

$$\mathbf{H}_P(x, y, z) = \det \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix}.$$

Remark 3.2.4. The second partial derivatives of $P(x, y, z)$ are homogeneous of degree $d - 2$, so $\mathbf{H}_P(x, y, z)$ is a homogeneous polynomial of degree $3(d - 2)$.

We also introduce a result that simplifies the calculation of the Hessian.

Lemma 3.2.5. Let $P(x, y, z)$ be a homogeneous polynomial of degree $d \geq 2$. Then

$$z^2 \mathbf{H}_P(x, y, z) = (d - 1)^2 \det \begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & dP/(d - 1) \end{pmatrix}$$

Proof. The proof is the same as the proof of lemma 3.30 of [1]. □

We can now introduce the points of inflection of a curve.

Definition 3.2.6. A point of inflection of a projective curve C defined by a polynomial $P(x, y, z)$ is a non-singular point $[a : b : c] \in C$ such that $\mathbf{H}_P(a, b, c) = 0$.

As we did with the points of intersection, we bound the points of inflection of a curve.

Remark 3.2.7. Every point in an irreducible projective curve C is a point of inflection if and only if C is a projective line.

Proposition 3.2.8. Let C be a non-singular projective curve of degree n.

1. If $n \geq 3$ then C has at least one point of inflection.
2. If $n \geq 2$ then C has at most $3n(n-2)$ points of inflection.

Proof. Let C be a non-singular projective curve of degree n defined by a polynomial $P(x, y, z)$. By remark 3.2.4, $\mathbf{H}_P(x, y, z)$ is a homogeneous polynomial of degree $3(n-2)$ and, therefore, when $n \geq 2$ it defines a projective curve D in the generalised sense of remark 2.1.3.

(1) follows from applying theorem 3.1.7 to C and D. If $n = 2$, however, $\mathbf{H}_P(x, y, z)$ is constant and non-zero, so C does not have a point of inflection. Thus, $n \geq 3$.

(2) follows from applying the weak version of Bézout's theorem (theorem 3.1.8) to C and D as long as they do not have a common irreducible factor. The curve C is non-singular, so by corollary 3.1.8.1 it is irreducible, and, therefore, C and D only have a common irreducible factor if C is contained in D. If C was contained in D, however, every point in C would be a point of inflection and, by remark 3.2.7, C would be a projective line, contradicting the assumption that $n \geq 2$. □

Finally, we present the equivalence of non-singular projective cubics under projective transformations.

Theorem 3.2.9. For every non-singular projective cubic C, there exists a $\lambda \in \mathbb{C} - \{0, 1\}$ such that C is equivalent under a projective transformation to the curve defined by $y^2 z = x(x-z)(x-\lambda z)$.

Proof. Let C be a non-singular projective cubic defined by a polynomial $P(x, y, z)$. The degree of C is 3, so by proposition 3.2.8 C has at least one point of inflection. We can apply a projective transformation that takes $[0 : 1 : 0]$ to a point of inflection of C and the projective line $z = 0$ to its tangent line to C, having

$$\mathbf{H}_P(0,1,0) = P(0,1,0) = \frac{\partial P}{\partial x}(0,1,0) = \frac{\partial P}{\partial y}(0,1,0) = 0.$$

Note that since C is non-singular, then $\frac{\partial P}{\partial z}(0,1,0) \neq 0$. Now we can apply lemma 3.2.5 (reversing the roles of y and z) to get

$$y^2 \mathbf{H}_P(x, y, z) = 4 \det \begin{pmatrix} P_{xx} & P_x & P_{xz} \\ P_x & \frac{3}{2}P & P_z \\ P_{zx} & P_z & P_{zz} \end{pmatrix},$$

which we can evaluate at $[0:1:0]$ to get

$$\mathbf{H}_P(0,1,0) = 0 = 4\det\begin{pmatrix} P_{xx} & 0 & P_{xz} \\ 0 & 0 & P_z \\ P_{zx} & P_z & P_{zz} \end{pmatrix} = -4(\frac{\partial P}{\partial z}(a,b,c))^2 \frac{\partial P}{\partial^2 x}(0,1,0).$$

Thus, $\frac{\partial P}{\partial^2 x}(x,y,z) = 0$ and, therefore, if $\phi(x,z)$ is a homogeneous polynomial of degree 3 and $\beta = \frac{\partial P}{\partial z}(x,y,z) \neq 0$, then we have that $P(x,y,z) = yz(\alpha x + \beta y + \gamma z) + \phi(x,z)$. Now we can apply the projective transformation $[x:y:z] \to [x : y + \frac{\alpha x + \gamma z}{2\beta} : z]$ so that, if $\psi(x,z)$ is a homogeneous polynomial of degree 3, then C is defined by the equation $\beta y^2 z + \psi(x,z) = 0$.

The curve C is non-singular, so by corollary 3.1.7.1 it is irreducible, and hence $\psi(x,z)$ is not divisible by z, so the coefficient of x^3 in $\psi(x,z)$ is not zero. The polynomial $\psi(x,z)$ is homogeneous of degree 3, so by corollary 2.3.2 it can be decomposed as the product of 3 linear factors. Thus, since C is non-singular, there must exist $a,b,c \in \mathbb{C}$ distinct such that we can apply a projective transformation that makes C be defined by the equation

$$y^2 z = (x - az)(x - bz)(x - cz).$$

Finally, let $\zeta \in \mathbb{C}$ satisfy $\zeta^2 = (b-a)^{-3}$. There must exist a $\lambda \in \mathbb{C} - \{0,1\}$ such that the projective transformation $[x:y:z] \to [\frac{x-az}{b-a} : \zeta y : z]$ takes the equation of C to $y^2 z = x(x-z)(x-\lambda z)$. □

Theorem 3.2.9 only applies to non-singular curves. The case for singular curves is more complicated since, under a projective transformation, they are not all equivalent to a single curve, but rather to either:

$$\begin{aligned} &y^2 z = x^3, \\ &y^2 z = x^2(x+z), \text{ or} \\ &y^2 z = x(x-z)(x-\lambda z). \end{aligned}$$

We will not prove this. Instead, we present an example to give an intuition as to why it is true.

Example 3.2.10. Let C be the singular projective curve defined by the equation $2xyz = x^2(9x - y + z) + y^2(-x + y + z)$. The projective transformation $[x:y:z] \to [-2x : x - y : x + y + z]$ takes the equation of C to the first case: $y^2 z = x^3$.

We finish the chapter with an example that illustrates another equivalence relation under projective transformations between non-singular projective curves.

Example 3.2.11. Let C be a non-singular projective curve defined by a homogeneous polynomial $P(x,y,z)$ of degree 3. We want to show that if p is a point of inflection of C then there is a projective transformation that takes p to the point $[0:1:0]$ and the equation of C to $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$.

As in the proof of theorem 3.2.9, we know that there is a projective transformation that takes the equation of C to $y^2z = (4x - az)(x - bz)(x - cz)$, which, after multiplying the right hand side out, becomes

$$y^2z = 4x^3 - (a + 4b + 4c)x^2z + (ab + ac + 4bc)xz^2 - abcz^3.$$

To get the expression we want, we let $g_1 = a+4b+4c$, $g_2 = ab+ac+4bc$, and $g_3 = abc$, and we apply the projective transformation $[x : y : z] \to [x : \sqrt{y^2 - g_1x^2} : z]$ to take the equation of C to $y^2z = 4x^3 - g_2xz^2 - g_3z^3$.

Chapter 4

Topology

4.1 The relation between plane curves and $\mathbb{P}^1_{\mathbb{C}}$

The next step in the study of projective curves is to study their topological structure. We will focus for the rest of this work on plane curves: non-singular projective curves in $\mathbb{P}^2_{\mathbb{C}}$.

We can study the topological structure of plane curves by relating them to projective lines. A projective line is homeomorphic to $\mathbb{P}^1_{\mathbb{C}}$ which, in its turn, is homeomorphic to the 2-dimmensional unit sphere in $\mathbb{R}^3$, the Riemann sphere. Thus, we can study the topology of plane curves by relating them to $\mathbb{P}^1_{\mathbb{C}}$.

Definition 4.1.1. Let C be a plane curve. The function $\phi : C \to \mathbb{P}^1_{\mathbb{C}}$ is a well-defined surjection such that $\phi([x : y : z]) = [x : z]$.

The function ϕ takes many points in a plane curve to a single point in $\mathbb{P}^1_{\mathbb{C}}$. But how many? The answer in general is the degree of the curve, but there are exceptions. There are some points that 'weight' more than others, points that count for more than just one. The ramification index measures this 'weight'.

Definition 4.1.2. Let C be a plane curve defined by a polynomial $P(x, y, z)$ of degree higher than one. The ramification index $v_\phi([a : b : c])$ of ϕ at each point $[a : b : c] \in C$ is the order of the zero of $P(a, y, c)$ at $y = b$.

The ramification points of ϕ are the points in C whose ramification index is higher than one, and their image is the branch locus of ϕ.

Remark 4.1.3. The choice of coordinate y in definition 4.1.2 is symmetric; the polynomial $P(x, y, z)$ is homogeneous, so we could have also chosen the x or z coordinate.

The following example illustrates how to calculate the ramification index of the points in a curve.

Example 4.1.4. Let C be the plane curve defined by the polynomial $P(x, y, z) = y^2 - xz$ and let $[a : b : c]$ be a point in C. We can factorize the single variable polynomial $P(a, y, c)$ to get $P(a, y, c) = y^2 - ac = (y + \sqrt{ac})(y - \sqrt{ac})$; and, since

$P(a, y, c)$ has a double root when a or c are 0, then $v_\phi[1 : 0 : 0] = 2 = v_\phi[0 : 0 : 1]$ and $v_\phi[a : b : c] = 1$ for every other point in C.

Remark 4.1.5. The ramification index of each point $[a : b : c]$ in a curve defined by a polynomial $P(x, y, z)$ is related to the partial derivatives with respect to y of the polynomial that generates the curve. In particular:

1. $v_\phi([a : b : c]) > 0 \iff P(a, b, c) = 0$,
2. $v_\phi([a : b : c]) > 1 \iff P(a, b, c) = \frac{\partial P}{\partial y}(a, b, c) = 0$, and
3. $v_\phi([a : b : c]) > 2 \iff P(a, b, c) = \frac{\partial P}{\partial y}(a, b, c) = \frac{\partial P}{\partial^2 y}(a, b, c) = 0$.

Again, the ramification index of a point $[a : b : c]$ measures the 'weight' of $[a : b : c]$: the higher the 'weight', the less points there will be in $\phi^{-1}([a : c])$. Lemma 4.1.6 formalizes this relation between the number of points in $\phi^{-1}([a : c])$ and their ramification index.

Lemma 4.1.6. The inverse image of ϕ at every $[a : c] \in \mathbb{P}^1_{\mathbb{C}}$ contains

$$d - \sum_{p \in \phi^{-1}([a:c])} (v_\phi(p) - 1)$$

points. In particular $\phi^{-1}([a : c])$ contains exactly d points if and only if it does not contain any ramification points of ϕ.

Proof. The proof can be found in [1] as the proof of lemma 4.5. □

The following example provides some intuition as to why the relation holds.

Example 4.1.7. Let C be the plane curve defined by the polynomial $P(x, y, z) = (y - \sqrt{xz})(y + \sqrt{xz})^2$. The function ϕ takes two points in C to the point $[1 : 1]$: the points $[1 : 1 : 1]$ and $[1 : -1 : 1]$; and since $P(1, y, 1) = (y-1)(y+1)^2$, the ramification index of ϕ at these two points is $v_\phi([1 : 1 : 1]) = 1$ and $v_\phi([1 : -1 : 1]) = 2$. We can now plug these values into the equality given by lemma 4.1.6 to see that – indeed – the relation holds: $3 - (v_\phi([1 : 1 : 1]) - 1) - (v_\phi([1 : -1 : 1]) - 1) = 2$.

The ramification index is the key to understanding how plane curves relate to $\mathbb{P}^1_{\mathbb{C}}$. As it stands now, however, it is too free. We need to bound it for it to be useful.

Lemma 4.1.8. There exists a suitable projective transformation for every plane curve C such that, for every point $[a : b : c] \in C$, the ramification index $v_\phi[a, b, c] \leq 2$.

Proof. Let C be a plane curve defined by a polynomial $P(x, y, z)$ and let $[a : b : c]$ be a point in C such that $\frac{\partial P}{\partial y}(a, b, c) = 0$. By remark 4.1.5 we have to prove that $\frac{\partial P}{\partial^2 y}(a, b, c) \neq 0$. By proposition 3.2.8, C has a finite number of inflection points, so we can apply a suitable projective transformation such that the point $[0 : 1 : 0]$ does not lie on C or on any of the tangent lines to C at its points of inflection. Since $[0 : 1 : 0]$ does not lie on C, the values a and c cannot both be zero. Let us assume that $c \neq 0$ (the case for $a \neq 0$ is equivalent). Similarly, since $[0:1:0]$ is contained in the tangent to C at $[a:b:c]$, the point $[a:b:c]$ cannot be a point of inflection.

By lemma 3.2.5 we know that

$$\mathbf{H}_P(a,b,c) = \frac{(d-1)^2}{c^2} \det \begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & 0 \\ P_x & 0 & 0 \end{pmatrix} = -\frac{(d-1)^2}{c^2}(\frac{\partial P}{\partial x}(a,b,c))^2 \frac{\partial P}{\partial^2 y}(a,b,c),$$

and, therefore, since $H_P(a,b,c) \neq 0$, $\frac{\partial P}{\partial^2 y}(a,b,c) \neq 0$. □

Thanks to this bound we can now determine exactly the number of ramification points of a plane curve.

Lemma 4.1.9. If every point in a plane curve C of degree d has a ramification index of at most 2, then ϕ has exactly $d(d-1)$ ramification points in C.

Proof. Let C be a plane curve of degree d defined by a polynomial $P(x,y,z)$ such that $[0:1:0] \notin C$ and that, for every point $[a:b:c] \in C$, $v_\phi([a:b:c]) \leq 2$. Let D be the curve defined by $\frac{\partial P}{\partial y}(x,y,z)$. The degree of D is $d-1$ since $P(0,1,0) \neq 0$.

By corollary 3.1.15.1, we can prove lemma 4.1.9 by showing that every point $[a:b:c] \in C \cap D$ is a non-singular point of D and that the tangent lines to C and D at $[a:b:c]$ are distinct. A point $[a:b:c] \in C \cap D$ cannot be a singularity of D since that would imply that $P(a,b,c) = 0 = \frac{\partial P}{\partial y}(a,b,c) = \frac{\partial P}{\partial^2 y}(a,b,c)$, and, by remark 4.1.5, that $v([a:b:c]) > 2$. Similarly, the tangent lines to C and D at a point $[a:b:c]$ cannot be the same since that would imply that $\frac{\partial P}{\partial^2 y}(a,b,c)$ is a scalar multiple of $\frac{\partial P}{\partial y}(a,b,c)$ and, again by remark 4.1.5, that $v([a:b:c]) > 2$. □

4.2 The topological structure of plane curves

The next step is to classify different plane curves with respect to their topological structure. Before diving into it, however, a note on the notation: $\#S$ denotes the number of elements in a finite set S; $\triangle$ denotes the standard triangle in $\mathbb{R}^2$, and $\triangle^0$ denotes its interior.

Definition 4.2.1. A triangulation of a plane curve C is given by:

- V: a finite, non-empty set of points called vertices;
- E: a finite, non-empty set of continuous maps $e : [0,1] \to C$ called edges;
- F: a finite, non-empty set of continuous maps $f : \triangle \to C$ called faces.

We can triangulate $\mathbb{P}^1_{\mathbb{C}}$ by giving a triangulation of the Riemann sphere; instead, however, we describe a triangulation of $\mathbb{P}^1_{\mathbb{C}}$ for a given set of vertices.

Lemma 4.2.2. For every set $\{p_1, ..., p_r\} \subset \mathbb{P}^1_{\mathbb{C}}$ of at least three distinct points, there exists a triangulation of $\mathbb{P}^1_{\mathbb{C}}$ with $3r-6$ edges, $2r-4$ faces, and with the points $p_1, ..., p_r$ as its vertices.

Proof. Let $r \geq 3$ be a positive integer and let $p_1, ..., p_r$ be r distinct points in $\mathbb{P}^1_{\mathbb{C}}$. We can prove lemma 4.2.2 by induction on r.

If $r = 3$, then there is a projective transformation that takes p_1 to 1, p_2 to $e^{\frac{2\pi}{3}i}$, and p_3 to $e^{\frac{4\pi}{3}i}$. Thus, the vertices of the triangulation are the points $V = \{1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}\}$. By lemma 4.2.2, the triangulation must have three edges and two faces. The three edges of the triangulation are the segments of the unit circle in $\mathbb{C}$ that connect the vertices; and the two faces are the interior of the unit circle in $\mathbb{C}$, which we will call F_1, and the exterior of the unit circle in $\mathbb{C}$ together with the point at infinity, which we will call F_2. The set F_1 is a face because there exists a homeomorphism from $\triangle$ to F_1; and the set F_2 is a face because the projective transformation $z \to \frac{1}{z}$ takes F_2 to F_1.

Now suppose that $r > 3$ and that we have a triangulation with $3(r-1)-6$ edges, $2(r-1)-4$ faces, and with the points p_1, ..., p_{r-1} as vertices. Let p_r be each point in $\mathbb{P}^1_{\mathbb{C}}$ distinct to p_1, ..., p_{r-1}. The point p_r can lie either in the interior of a face or on an edge. If p_r lies in the interior of a face, we can make a new triangulation by connecting it to the vertices of the face; to do this we add three edges and we split the previous face into three. If p_r lies on an edge, we can make a new triangulation by connecting it to the two vertices of the faces the edge splits that the edge does not connect; to do this we split the previous edge into two, we had two new edges, and we split the previous two faces into four. □

Next we use the relation that ϕ establishes between a plane curve C and $\mathbb{P}^1_{\mathbb{C}}$ to generate a triangulation of C from a triangulation of $\mathbb{P}^1_{\mathbb{C}}$.

Proposition 4.2.3. Let C be a plane curve of degree d such that the point $[0:1:0] \notin C$, let V be a set of points in C that contains the branch locus of ϕ, and let (V, E, F) be a triangulation of $\mathbb{P}^1_{\mathbb{C}}$. There exists a triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ of C such that

$$\begin{aligned}
&\tilde{V} = \phi^{-1}(V),\\
&\tilde{E} = \{\tilde{e} : [0,1] \to C : \tilde{e} \text{ continuous}, \phi \text{ o } \tilde{e} \in E\},\\
&\tilde{F} = \{\tilde{f} : \triangle \to C : \tilde{f} \text{ continuous}, \phi \text{ o } \tilde{f} \in E\}.
\end{aligned}$$

Furthermore:

$$\begin{aligned}
&\#\tilde{V} = d\#V - \sum_{p \in R}(v_\phi(p) - 1),\\
&\#\tilde{E} = d\#E,\\
&\#\tilde{F} = d\#F.
\end{aligned}$$

Proof. The proof can be found in [1] as the proof of proposition 4.22. □

Theorem 4.2.4 combines lemma 4.2.2 and proposition 4.2.3 to define a triangulation of a plane curve in terms of the degree of the curve.

Theorem 4.2.4. Let C be a plane curve of degree d. For every natural number $r \geq 3$ such that $r \geq d(d-1)$, the curve C has a triangulation with $dr - d(d-1)$ vertices, $3(r-2)d$ edges, and $2(r-2)d$ faces.

Proof. Let C be a plane curve of degree d, let r be a positive integer such that $r \geq 3$ and $r \geq d(d-1)$, and let V be a set of r points in C that contains the branch locus of ϕ. By lemma 4.2.2, there exists a triangulation (V, E, F) of $\mathbb{P}^1_{\mathbb{C}}$ such that $\#V = r, \#E = 3r-6, \#F = 2r-4$. By proposition 4.2.3, there exists another triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ of $\mathbb{P}^1_{\mathbb{C}}$ such that

$$\begin{aligned} \#\tilde{V} &= d\#V - \sum_{p \in R}(v_\phi(p) - 1), \\ \#\tilde{E} &= d\#E = d(3r-6), \\ \#\tilde{F} &= d\#F = d(2r-4); \end{aligned}$$

and since, by lemmas 4.1.8 and 4.1.9, the function ϕ has exactly $d(d-1)$ ramification points in C with a ramification index of two, $\#\tilde{V} = dr - d(d-1)$.

□

Triangulations let us define the Euler characteristic and the genus of a curve and, thus, they let us classify the different topological structures.

Definition 4.2.5. Let C be a plane curve. The Euler characteristic of a triangulation of C is $\mathcal{X}(C) = \#V - \#E + \#F$, and the genus of C is $g = \frac{1}{2}(2 - \mathcal{X}(C))$.

We now have the tools to establish a relation between the degree and the genus of a curve.

Corollary 4.2.5.1 (The degree-genus formula)**.** The degree d of a plane curve C determines the Euler characteristic, $\mathcal{X}(C) = d(3-d)$, and the genus, $g = \frac{1}{2}(d-1)(d-2)$, of C.

Proof. Let C be a plane curve of degree d. By theorem 4.2.4 the Euler characteristic of C is $\mathcal{X}(C) = dr - d(d-1) - 3(r-2)d + 2(r-2)d = d(3-d)$, and by definition 4.2.5 the genus of C is $g = \frac{1}{2}(2 - d(3-d)) = \frac{1}{2}(d-1)(d-2)$. □

4.3 Examples

Example 4.3.1. We can triangulate a sphere with three vertices, three edges, and two faces. The Euler characteristic of the sphere is then 2 and, by definition 4.2.5, its genus is 0. If we set $g = 0$ in the degree-genus formula, then $d = 1$ or

$d = 2$. Thus, a plane line and a plane conic in $\mathbb{P}^2_{\mathbb{C}}$ are both topologically equivalent to a sphere.

Example 4.3.2. Figure 4.1 illustrates a triangulation of the torus with one vertex, three edges, and two faces. The Euler characteristic of the torus is then 0 and, by definition 4.2.5, its genus is 1. If we set $g = 1$ in the degree-genus formula, then $d = 3$. Thus, a plane cubic in $\mathbb{P}^2_{\mathbb{C}}$ is topologically equivalent to a torus.

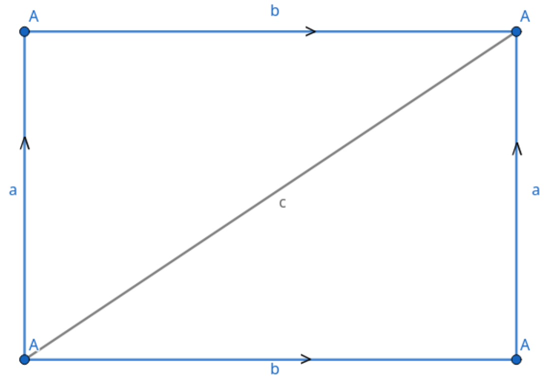

Figure 4.1: A triangulation of the torus

Chapter 5

Complex Analysis

5.1 Riemann surfaces

Many complex functions associate several outputs to the same input. The traditional theory of Complex Analysis solves this by introducing branches. A more elegant solution, however, is to detach Complex Analysis from the complex plane. Thus, we introduce surfaces.

Definition 5.1.1. A surface is a Hausdorff topological space S locally homeomorphic to $\mathbb{C}$ (that is: every point in S has an open neighborhood in S that is homeomorphic to an open subset of $\mathbb{C}$).

We can relate an open subset of a surface to an open subset of $\mathbb{C}$ with a homeomorphism called a chart. A collection of charts that cover a surface is called an atlas. An atlas, however, can have two conflicting charts, two charts that take the same region in a surface to different regions in $\mathbb{C}$. To fix this, we need to study how charts interact between each other.

Definition 5.1.2. A transition function of an atlas $\phi = \{\phi_n : U_n \to V_n : n \geq 1\}$ is a homeomorphism $\phi_{nm} = \phi_n \circ \phi_m^{-1} : \phi_n(U_n \cap U_m) \to \phi_n(U_n \cap U_m)$ between open subsets of $\mathbb{C}$.

Transition functions let us filter out the atlases with conflicting charts. An atlas without conflicting charts, an atlas whose transition functions are all holomorphic, is said to be holomorphic. A surface, however, can have many different holomorphic atlases. When are two holomorphic atlases equivalent? Two holomorphic atlases ϕ and ψ on a surface S are equivalent if the identity map $1_S : S \to S$ is holomorphic both as $\psi \circ 1_S \circ \phi^{-1}$ and as $\phi \circ 1_S \circ \psi^{-1}$. An equivalence class of holomorphic atlases gives a surface an analytical structure, a structure in which we can do Complex Analysis.

Definition 5.1.3. A Riemann surface is a surface S together with an equivalence class of holomorphic atlases on S.

The first step to do Complex Analysis on Riemann surfaces is to define how holomorphic functions operate on them with respect to an atlas.

Definition 5.1.4. Let S be a Riemann surface with a holomorphic atlas $\phi = \{\phi_n : U_n \to V_n : n \geq 1\}$. A continuous map $f : S \to \mathbb{C}$ is holomorphic with respect to ϕ at $x \in S$ if ϕ has a chart $\phi_n : U_n \to V_n$ such that $x \in U_n$ and $f \circ \phi_n^{-1} : V_n \to \mathbb{C}$ is holomorphic at $\phi_n(x)$.

The next step is to introduce holomorphic functions between two surfaces with respect to two of their atlases.

Definition 5.1.5. Let S and T be two Riemann surfaces with holomorphic atlases ϕ and ψ. A continuous map $f : S \to T$ is holomorphic with respect to ϕ and ψ if, for every chart $\phi_n : U_n \to V_n$ in ϕ and for every chart $\psi_m : W_m \to Y_m$ in ψ,

$$\psi_m \circ f \circ \phi_n^{-1} : \phi_n(U_n \cap f^{-1}(W_m)) \to Y_m$$

is holomorphic.

We can also define holomorphic functions between Riemann surfaces independently of the choice of atlas.

Definition 5.1.6. A continuous map $f : S \to T$ between two Riemann surfaces $(S, \mathbf{H})$ and $(T, \mathcal{F})$ is holomorphic if it is holomorphic with respect to a holomorphic atlas $\phi \in \mathbf{H}$ on S and $\psi \in \mathcal{F}$ on T.

Finally, we note that two Riemann surfaces S and T are biholomorphic, equivalent to each other, if there is a holomorphic bijection $f : S \to T$ (whose inverse is holomorphic).

5.2 Functions on Riemann surfaces

Definition 5.2.1. Let $\phi : U \to V$ be a holomorphic chart on an open subset U of a Riemann surface S. A piecewise-smooth path in S is a continuous map $\gamma : [a, b] \to S$ such that if $[c, d] \subseteq \gamma^{-1}(U)$, then $\phi \circ \gamma : [c, d] \to V$ is a piecewise-smooth path in the open subset V of $\mathbb{C}$.

Lemma 5.2.2. For every set of points $\{p_1, \ldots, p_r, q_1, \ldots, q_s\} \subseteq \mathbb{C}$ such that $r \geq 3$, there exists a triangulation (V, E, F) of $\mathbb{P}^1_{\mathbb{C}}$ such that $V = \{p_1, \ldots, p_r\}$ and the set $\{q_1, \ldots, q_r\}$ is contained in the interior of a face.

Proof. The proof is similar to the proof of lemma 4.2.2. It can be found in [1] as the proof of lemma 6.47. □

The study of meromorphic functions is central to the study of Complex Analysis. Thus, we define meromorphic functions on Riemann surfaces.

Definition 5.2.3. A meromorphic function on a Riemann surface S is a holomorphic function $f : S \to \mathbb{P}^1_{\mathbb{C}}$ that is not ∞ on any connected component of S.

Let $\phi = \{\phi_n : U_n \to V_n : n \geq 1\}$ be the atlas of a Riemann surface S. How can we differentiate a meromorphic function g on S? We can differentiate all the meromorphic functions $g \circ \phi_n^{-1} : V_n \to \mathbb{C}$ and transfer them back to S, to get the meromorphic functions $(g \circ \phi_n^{-1})' \circ \phi_n$ on the open subsets U_n that cover S. The result, however, is not a function on S, but rather an abstract object: a (meromorphic) differential, written dg. We can multiply dg by a meromorphic function f on S to get a new differential, written $f\,dg$. Furthermore, two meromorphic differentials $f\,dg$ and $\tilde{f}\,d\tilde{g}$ on S are the same if all the meromorphic functions $(f \circ \phi_n^{-1})(g \circ \phi_n^{-1})'$ and $(\tilde{f} \circ \phi_n^{-1})(\tilde{g} \circ \phi_n^{-1})'$ that represent them on U_n are the same.

Definition 5.2.4. Let $\phi = \{\phi_n : U_n \to V_n : n \geq 1\}$ be an atlas of a Riemann surface S. A meromorphic differential $f\,dg$ on S is an equivalence class of pairs of meromorphic functions on S such that two pairs (f, g) and $(\tilde{f}, \tilde{g})$ are equivalent if and only if, for every z in V,

$$(f \circ \phi^{-1})(z) \cdot (g \circ \phi^{-1})'(z) = (\tilde{f} \circ \phi^{-1})(z) \cdot (\tilde{g} \circ \phi^{-1})'(z).$$

Remark 5.2.5. If z is the identity function on $\mathbb{C}$, then every meromorphic differential $f\,dg$ on $\mathbb{C}$ can be expressed uniquely as $(f \circ g')\,dz$, and, therefore, a meromorphic differential on $\mathbb{C}$ is equivalent to a meromorphic function on $\mathbb{C}$.

As in the complex plane, meromorphic differentials have zeros and poles.

Definition 5.2.6. Let S be a Riemann surface, and let $\phi : U \to V$ be a holomorphic chart on an open neighbourhood U of a point $p \in S$. A meromorphic differential $f\,dg$ has a pole at p if the meromorphic function $(f \circ \phi^{-1})(g \circ \phi^{-1})'$ has a pole at $\phi(p)$. If $f\,dg$ does not have a pole, however, we say $f\,dg$ is a holomorphic differential.

A meromorphic differential has different properties as the differential of a meromorphic function. For example:

Lemma 5.2.7. If a meromorphic differential on a plane curve only has one pole, then that pole is not a simple pole.

Proof. Let $\omega = g\,dh$ be a meromorphic differential on a plane curve C with just one pole, at q. We are going to show that q is not a simple pole.

We start by choosing coordinates in $\mathbb{P}^2_{\mathbb{C}}$ such that the point $[0 : 1 : 0] \notin C$ and such that 0, $\phi(q)$, and ∞ are distinct and are not branch points of ϕ. By lemma 5.2.2, there exists a triangulation (V, E, F) such that the set V contains the branch points of ϕ, a face $f_0 \in F$ contains 0 and $\phi(q)$, and another face $f_\infty \in F$ contains ∞. By proposition 4.2.3, there exists a triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ such that $\tilde{V} = \phi^{-1}(V)$, $\tilde{E} = \{\tilde{e} : \phi \circ \tilde{e} \in E\}$, and $\tilde{F} = \{\tilde{f} : \phi \circ \tilde{f} \in F\}$. Furthermore, we can subdivide the triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ so that each face has at most one branch point as a vertex.

Thus, let $\tilde{f} \in \tilde{F}$ and $f = \phi \circ \tilde{f} \in F$. If $f = f_\infty$, then $\phi : \tilde{f}(\triangle) \to f(\triangle)$ is a homeomorphism whose restriction to $\tilde{f}(\triangle - \{(0,0), (0,1), (1,0)\})$ of C is the restriction of a holomorphic chart on C; if $f \neq f_\infty$, however, we must compose ϕ with the map $z \to \frac{1}{z}$.

Let σ_1, σ_2, and σ_3 be the maps along the three edges of $\triangle$, and let $\tilde{\gamma}_{\tilde{f}} = \tilde{f} \circ \sigma_1 + \tilde{f} \circ \sigma_2 + \tilde{f} \circ \sigma_3$. Then $\gamma_{\tilde{f}} = \phi \circ \tilde{\gamma}_{\tilde{f}}$ is a closed piecewise-smooth path in $\mathbb{P}^1_{\mathbb{C}}$ whose image is the boundary of $f(\triangle)$ and, by definition 5.2.1,

$$\int_{\tilde{\gamma}_{\tilde{f}}} \omega = \int_{\gamma_{\tilde{f}}} (\phi|^{-1}_{\tilde{f}(\triangle)}) * \omega = \int_{\gamma_{\tilde{f}}} (g \circ \phi|^{-1}_{\tilde{f}(\triangle)})(h \circ \phi|^{-1}_{\tilde{f}(\triangle)})'(z)dz.$$

If $q \in \tilde{f}(\triangle^0)$, then $f = f_0$ and $(g \circ \phi|^{-1}_{\tilde{f}(\triangle)})(h \circ \phi|^{-1}_{\tilde{f}(\triangle)})'(z)$ has a simple pole at $\phi(q)$ inside γ and no other poles, and since the residue of a meromorphic function at a simple pole in $\mathbb{C}$ is always nonzero, then by Cauchy's residue theorem $\int_{\tilde{\gamma}_{\tilde{f}}} \omega \neq 0$. If $q \notin \tilde{f}(\triangle^0)$, however, then $(g \circ \phi|^{-1}_{\tilde{f}(\triangle)})(h \circ \phi|^{-1}_{\tilde{f}(\triangle)})'(z)$ does not have a pole in γ and $\int_{\tilde{\gamma}_{\tilde{f}}} \omega = 0$. The point q is then in $\tilde{f}(\triangle^0)$ for just one face of $\tilde{F}$, and if we sum over all the faces $\tilde{f} \in \tilde{F}$ we get $\sum_{\tilde{f} \in \tilde{F}} \int_{\tilde{\gamma}_{\tilde{f}}} \omega \neq 0$.

Each of the integrals $\int_{\tilde{\gamma}_{\tilde{f}}} \omega$ can be expressed as the sum of the integrals along the three edges $e^1_{\tilde{f}}$, $e^2_{\tilde{f}}$, and $e^3_{\tilde{f}}$. The map σ_i, however, goes in a different direction for the two faces of every edge $e \in E$ and, thus, the integrals of ω along the edges cancel in pairs and $\sum_{\tilde{f} \in \tilde{F}} \int_{\tilde{\gamma}_{\tilde{f}}} \omega = 0$.

This contradiction proves lemma 5.2.7. □

5.3 The relation between plane curves and Riemann surfaces

To study how a function on a plane curve behaves, we introduce divisors.

Definition 5.3.1. A divisor D on a plane curve C is a formal sum $D = \sum_{p \in C} n_p\, p$ such that n_p is only zero for a finite set of points in C. Furthermore, the degree of D is $\deg D = \sum_{p \in C} n_p$.

Lemma 5.3.2. For every divisor D on a plane curve C and for every natural number m_0, there exists a natural number $m \geq m_0$ and points p_1, ..., p_k in C such that

$$D + p_1 + ... + p_k \sim m \sum_{p \in C} \mathbf{I}_p(C, L)\, p.$$

Proof. Let C be a plane curve, let D be a divisor on C, let H be a divisor on C such that $H = \sum_{p \in C} \mathbf{I}_p(C, L)\, p$, and let m_0 be a natural number. We start by adding points to D so that $\deg D \geq m_0$ and that $n_p \geq 0$ for every $p \in C$.

The ratio of two linear homogeneous polynomials defines a meromorphic function on C, so if a line in $\mathbb{P}^2_{\mathbb{C}}$ intersects C at d distinct points q_1, ..., q_d, then $q_1 + ... + q_d \sim H$. For each point in p we can find a line in $\mathbb{P}^2_{\mathbb{C}}$ that contains p and that intersects C at d distinct points: $q^{(p)}_1 = p, q^{(p)}_2 ..., q^{(p)}_d$. Thus, if $m = \deg D$, then $m \geq m_0$ and

$$mH \sim \sum_{n_p>0} n_p(p + \sum_{i=2}^{d} q_i^{(p)}) = D + p_1 + ... + p_k.$$

□

The zeros and poles of meromorphic functions not only hold information about the function, but about the curve they are in. A principal divisor is a way of organizing these zeros and poles.

Definition 5.3.3. A principal divisor is a divisor of a non-zero meromorphic function f such that: $n_p = m$ if f has a zero of multiplicity m at p, $n_p = -m$ if f has a pole of multiplicity m at p, and $n_p = 0$ otherwise.

Definition 5.3.4. A canonical divisor is a divisor of a meromorphic differential.

Every non-zero meromorphic function on C has the same number of zeros and poles, counted with multiplicty. In terms of divisors:

Proposition 5.3.5. The degree of every principal divisor on a plane curve is zero.

Proof. Let g be a meromorphic function on a plane curve C with zeros and poles at the points q_1, ..., q_t. We want to show that g has the same number of zeros and poles up to multiplicity.

The meromorphic differential dg/g on C has poles at the points q_1, ..., q_t.

As in lemma 5.2.7, we start by choosing coordinates in $\mathbb{P}^2_{\mathbb{C}}$ such that the point $[0 : 1 : 0] \notin C$ and such that 0, $\phi(q_1)$, ..., $\phi(q_t)$, and ∞ are distinct and are not branch points of ϕ. By lemma 5.2.2, there exists a triangulation (V, E, F) such that the set V contains the branch points of ϕ, a face $f_0 \in F$ contains 0, $\phi(q_1)$, ..., $\phi(q_t)$, and another face $f_\infty \in F$ contains ∞. Furthermore, let σ_1, σ_2, and σ_3 be the maps along the edges of $\triangle$, let $\tilde{\gamma}_{\tilde{f}} = \tilde{f} \circ \sigma_1 + \tilde{f} \circ \sigma_2 + \tilde{f} \circ \sigma_3$, and let $\gamma_{\tilde{f}} = \phi \circ \tilde{\gamma}_{\tilde{f}}$. There exists a triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ of C such that

$$\sum_{\tilde{f} \in \tilde{F}} \int_{\tilde{\gamma}_{\tilde{f}}} \frac{dg}{g} = \sum_{\tilde{f} \in \tilde{F}} \int_{\gamma} \frac{(g \circ \phi|^{-1}_{\tilde{f}(\triangle)})'(z)}{(g \circ \phi|^{-1}_{\tilde{f}(\triangle)})(z)} dz = 0,$$

and the function dg/g has a pole at a point $a \in \gamma_{\tilde{f}}$ if and only if $g \circ \phi|^{-1}_{\tilde{f}(\triangle)}$ has either a zero at a with multiplicity ρ or a pole at a with multiplicity $-\rho$. The restriction of ϕ to $\tilde{f}(\triangle^0)$ is, however, a holomorphic chart on C, so g either has a zero at $\phi|^{-1}_{\tilde{f}(\triangle^0)}(a)$ with multiplicity ρ or a pole at $\phi|^{-1}_{\tilde{f}(\triangle^0)}$ with multiplicity $-\rho$. Thus, by Cauchy's residue theorem, if $Z(\tilde{f})$ and $P(\tilde{f})$ are the sum of the zeros and the poles of g in the interior of the face $\tilde{f}$ (both counted with multiplicity), then $\int_{\tilde{\gamma}_{\tilde{f}}} dg/g = \pm(Z(\tilde{f}) - P(\tilde{f})$.

All the zeros and poles of g lie in $\phi^{-1}(f_0(\triangle^0))$. Furthermore, the sign of $\int_{\tilde{\gamma}} dg/g$ is consistent for every $\tilde{f} \in \tilde{F}$ such that $\phi \circ \tilde{f} = f_0$ and, therefore,

$$\sum_{\tilde{f}\in\tilde{F}:\phi\circ\tilde{f}=f_0}\int_{\tilde{\gamma}_{\tilde{f}}}\frac{dg}{g}+\sum_{\tilde{f}\in\tilde{F}:\phi\circ\tilde{f}\neq f_0}\int_{\tilde{\gamma}_{\tilde{f}}}\frac{dg}{g}=\pm(Z-P)+0\implies Z-P=0.$$

□

Two divisors D_1 and D_2 are linearly equivalent, written $D_1\sim D_2$, if D_1-D_2 is a principal divisor.

Corollary 5.3.5.1. If two divisors on a plane curve are linearly equivalent, then they have the same degree.

Proof. Corollary 5.3.5.1 follows immidiately from proposition 5.3.5. □

By proposition 5.3.5, the degree of every principal divisor is zero, not saying much about the curve they are on. The degree of a canonical divisor, however, is linked to the topological structure of the curve it is on.

Proposition 5.3.6. The degree of every canonical divisor on a plane curve of genus g is $2g-2$.

Proof. Let C be a plane curve of genus g defined by a polynomial $P(x,y,z)$. By proposition 5.3.5 we just have to show that there exists a non-zero meromorphic differential ω on C such that $\deg(\omega)=2g-2$.

We start by choosing coordinates such that the point $[0:1:0]\notin C$. By corollary 3.1.7.1 the polynomial $P(x,y,z)$ is irreducible, and by the Weak form of Bézout's theorem there are only finitely many points $[a:b:c]\in C$ such that $\frac{\partial P}{\partial y}(a,b,c)=0$. Furthermore, since $[0:1:0]\notin C$, the coordinates a and c of those points cannot both be zero; and, thus, after the projective transformation $(x,y,z)\to(x,y,\alpha x+z)$, the coordinate c of those points is not zero.

Let ω be the meromorphic differential $d(x/z)$ of the meromorphic function x/z on C. First, let us consider the points $[a:b:c]\in C$ such that $\frac{\partial P}{\partial y}(a,b,c)\neq 0$. If $c\neq 0$, then the function x/z can be thought of as a holomorphic chart near $[a:b:c]$ on C and, thus, does not have zeros or poles. If $c=0$, then $a\neq 0$; so if $v=z/x$, then $\omega=d(1/v)=-v^{-2}dv$ and, thus, ω has a pole of multiplicity two. Moreover, if $c=0$, then there are no points in C where the line $z=0$ is tangent to C and, with it, by corollary 3.1.15.1 the line $z=0$ intersects C at d distinct points. These points contribute $-2d$ to the degree of the divisor (ω).

Finally, let us consider the points $[a:b:c]\in C$ such that $\frac{\partial P}{\partial y}(a,b,c)=0$. At these points, $c\neq 0$ by our choice of coordinates and $\frac{\partial P}{\partial x}(a,b,c)\neq 0$, since that would imply by Euler's relation that $\frac{\partial P}{\partial z}(a,b,c)=0$ and, with it, that $[a:b:c]$ is a singularity of C. Thus, the function $u=y/z$ is a holomorphic chart on C near these points, and locally x/z is a holomorphic function $f(u)$ of u such that $P(f(u),u,1)=0$. After differentiating m times the identity $P(f(u),u,1)=0$ we can see that if $f^{(k)}(u_0)=0$ for a natural number $k<m$, then

$$f^{(m)}(u_0) = -\frac{P_y^{(m)}(u_0, f(u_0), 1)}{\frac{\partial P}{\partial x}(u_0, f(u_0), 1)}.$$

Thus, the smallest natural number m such that $f^{(m)}(u_0) \neq 0$ is the same as the smallest natural number m such that $P_y^{(m)}(u_0, f(u_0), 1) \neq 0$.

Furthermore, by remark 4.1.5, these points are the ramification points of the function ϕ and therefore, since $\omega = d(f(u)) = f'(u)du$, the multiplicity of a zero of ω at them is $v_\phi([a : b : c]) - 1$. By lemmas 4.1.8 and 4.1.9, we can choose coordinates so that ϕ has exactly $d(d-1)$ ramification points and, at them, $v_\phi([a : b : c]) - 1 = 1$. Thus, these points contribute $d(d-1)$ to the degree of the divisor (ω).

The degree of ω is then $deg(\omega) = d(d-1) - 2d = d(d-3)$ and, by the degree-genus formula, $deg(\omega) = 2g - 2$. □

By corollary 5.3.5.1, principal divisors on a plane curve C link divisors of the same degree on C. Canonical divisors are also linked to divisors on C. To study this link, we introduce the vector space $\mathcal{L}(D)$.

Definition 5.3.7. Let D be a divisor on a plane curve C. The vector space $\mathcal{L}(D)$ is the set of meromorphic functions f on C such that $(f) + D \geq 0$, together with the zero function. Equivalently: a meromorphic function f belongs to $\mathcal{L}(D)$ if f is holomorphic at every point $p \in C$ except at the points for which $n_p > 0$, where f has a pole of order at most n_p, and for which $n_p < 0$, where f has a zero of order at least $-n_p$.

The dimension of $\mathcal{L}(D)$ is noted $l(D)$, and it is also linked by principal divisors.

Lemma 5.3.8. If D_1 and D_2 are two divisors on a plane curve such that $D_1 \sim D_2$, then $l(D_1) = l(D_2)$.

Proof. Let D_1 and D_2 be two divisors on a plane curve C. If $D_1 \sim D_2$, then the divisor $D_1 - D_2$ is a principal divisor corresponding to a meromorphic function g. Moreover, the map $f \to fg$ defines an isomorphism between D_1 and D_2, so the dimensions of $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ must be the same. □

The dimension of a vector space $\mathcal{L}(D)$, however, reveals information about the structure of the curve C that D is on. For instance, we can bound the sum of the dimensions of several vector spaces for every point in C.

Lemma 5.3.9. Let κ be a canonical divisor on a plane curve C of genus g. For every divisor D on C and for every point p in C,

$$0 \leq l(D + p) - l(\kappa - D - p) - l(D) + l(\kappa - D) \leq 1.$$

Proof. Let C be a plane curve, let D be a divisor on C, let $\kappa = (\omega)$ be a canonical divisor on C, and let p be a point in C. If there exists a meromorphic function f on C such that $(f) + D + p = 0$ at p and $(f) + D + p > 0$ at any other point, then

$\mathcal{L}(D)$ is contained in $\mathcal{L}(D+p)$ and $l(D+p)-l(D)=1$; otherwise, $\mathcal{L}(D)=\mathcal{L}(D+p)$. Thus, $0 \leq l(D+p)-l(D) \leq 1$ and, similarly, $0 \leq l(\kappa-D)-l(\kappa-D-p) \leq 1$.

To finish the proof we just have to show that $l(D+p)-l(D)$ and $l(\kappa-D)-l(\kappa-D-p)$ cannot both be one. If so, then there would exist a meromorphic function f on C such that $(f)+D+p=0$ at p and $(f)+D+p>0$ at any other point, and a meromorphic function g on C such that $(g)+\kappa-D=0$ at p and $(g)+\kappa-D>0$ at any other point. The meromorphic differential $fg\,\omega=(f)+(g)+\kappa \geq -p$ would then just have one pole of order one at p, contradicting lemma 5.2.7. □

We now introduce a special divisor H that links points in a plane curve C with the intersection multiplicity between C and a line in $\mathbb{P}^2_{\mathbb{C}}$. Furthermore, we give a bound for its vector space.

Lemma 5.3.10. Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$, let C be a plane curve of genus g, let κ be a canonical divisor on C, and let H be a divisor on a plane curve C such that $H=\sum_{p\in C}\mathbf{I}_p(C,L)\,p$. There exists a natural number m_0 such that if $m \geq m_0$, then

$$l(mH)-l(\kappa-mH) \geq 1-g+\deg(mH).$$

Proof. The proof can be found in [1] as the first part of example 6.40. □

Finally, we conclude this chapter with two results that link the dimension of a vector space of a divisor on a curve with the topological structure of the curve. The first result, Riemann's theorem, gives a lower bound; and the second result, Riemann-Roch's theorem, turns the lower bound into an equality.

Theorem 5.3.11 (Riemann's theorem)**.** For every divisor D on a plane curve of genus g with a canonical divisor κ,

$$l(D)-l(\kappa-D) \geq 1-g+\deg D.$$

Proof. Let D be a divisor on a plane curve C of genus g. By lemma 5.3.2 there exist points p_1, ..., p_k and a natural number $m \geq m_0$ such that $D+p_1+...+p_k \sim mH$, and by lemma 5.3.9 and induction on k we get the inequality

$$l(D)+l(\kappa-D) \geq l(D+p_1+...+p_k)-l(k-D-p_1-...-p_k)-k. \tag{5.1}$$

We will now combine different results to get the inequality in theorem 5.3.11. Firstly, by lemma 5.3.8 we know that $l(mH)-l(\kappa-mH)=l(D+p_1+...+p_k)-l(k-D-p_1-...-p_k)$, so equation (5.1) becomes

$$l(D)+l(\kappa-D) \geq l(mH)-l(\kappa-mH)-k. \tag{5.2}$$

Secondly, by lemma 5.3.10 we know that there exists a natural number m_0 such that if $m \geq m_0$, then $l(mH)-l(\kappa-mH) \geq 1-g+\deg mH$, so equation (5.2) becomes

$$l(D) + l(\kappa - D) \geq 1 - g + \deg(mH) - k. \tag{5.3}$$

Finally, by corollary 5.3.5.1 we know that $\deg(mH) = \deg(D + p_1 + ... + p_k) = \deg(D) + k$, so equation (5.3) becomes

$$l(D) + l(\kappa - D) \geq 1 - g + \deg(D),$$

as required.

□

Theorem 5.3.12 (Riemann-Roch). For every divisor D on a plane curve of genus g with a canonical divisor κ,

$$l(D) - l(\kappa - D) = 1 - g + \deg D.$$

Proof. Let D be a plane curve of genus g with a canonical divisor κ. We want to calculate the value of $l(D) - l(\kappa - D)$. We can apply theorem 5.3.11 to D to get the lower bound $l(D) - l(\kappa - D) \geq 1 - g + \deg D$ and to $\kappa - D$ to get the upper bound $l(D) - l(\kappa - D) \leq -1 + g - \deg(\kappa - D)$. By proposition 5.3.6, the degree of κ is $2g - 2$; thus $l(D) - l(\kappa - D) \leq 1 - g + \deg D$ and, with it, the lower bound and the upper bound are the same. □

5.4 Examples: divisors on $\mathbb{P}^1_{\mathbb{C}}$

The divisors on $\mathbb{P}^1_{\mathbb{C}}$ are scarce. Two divisors on $\mathbb{P}^1_{\mathbb{C}}$ are linearly equivalent if their difference is principal and, therefore, the quotient of the set of divisors on $\mathbb{P}^1_{\mathbb{C}}$ by the set of principal divisors on $\mathbb{P}^1_{\mathbb{C}}$ (called the Weil divisor class group $CI(\mathbb{P}^1_{\mathbb{C}})$) is the set of divisors modulo linear equivalence. By corollary 5.3.5.1, if two divisors are linearly equivalent, then they have the same degree. Thus, $CI(\mathbb{P}^1_{\mathbb{C}})$ is isomorphic to $\mathbb{Z}$ and there are $\mathbb{Z}$ non linearly equivalent divisors on $\mathbb{P}^1_{\mathbb{C}}$.

The divisors on $CI(\mathbb{P}^1_{\mathbb{C}})$ also have special properties. For example: if the degree of a divisor $D = \sum_{p \in \mathbb{P}^1_{\mathbb{C}}} n_p\, p$ on $\mathbb{P}^1_{\mathbb{C}}$ is 0, then D is principal. To see why, consider that each point $p = [a : b] \in \mathbb{P}^1_{\mathbb{C}}$ has a corresponding homogeneous linear factor $f_p^{n_p} = (bx - ay)^{n_p}$, and since the sum of all the n_p values is zero, the product of all the $f_p^{n_p}$ factors is a meromorphic function that, at each point $p \in \mathbb{P}^1_{\mathbb{C}}$, is holomorphic if $n_p = 0$, has a zero of multiplicity n_p if $n_p > 0$, and has a pole of multiplicity n_p if $n_p < 0$.

We finish the chapter by giving two examples of the functions that define principal divisors on $\mathbb{P}^1_{\mathbb{C}}$. In example 5.4.1, we illustrate how a homogeneous polynomial of degree n in $\mathbb{P}^1_{\mathbb{C}}$ defines a principal divisor of degree n on $\mathbb{P}^1_{\mathbb{C}}$; and in example 5.4.2, we illustrate how, if $P(x, y)$ is a homogeneous polynomial of degree d, then the meromorphic function $g(x, y) = 1/P(x, y)$ defines a principal divisor of degree $-n$.

Example 5.4.1. The homogeneous polynomial $P(x, y) = x^3 - 2xy^2 + y^3$ of degree 3 in $\mathbb{P}^1_{\mathbb{C}}$ is zero at three points: $[1 : 1]$ and $[\pm\sqrt{5} - 1 : 2]$. The sum of these points forms a divisor on $\mathbb{P}^1_{\mathbb{C}}$ of degree 3.

Example 5.4.2. The meromorphic function $f(x, y) = 1/(x^3 - 2xy^2 + y^3)$ has three simple poles at the points $[1 : 1]$ and $[\pm\sqrt{5} - 1 : 2]$, and defines a principal divisor D such that $D = -[1 : 1] - [\sqrt{5} - 1 : 2] - [-\sqrt{5} - 1 : 2]$ of degree -3.

Chapter 6

Conclusion

In the first half of this end-of-degree project we defined affine curves (algebraic curves in $\mathbb{C}^2$) as the sets of zeros of nonconstant polynomials in two variables with no repeated factors. We defined their degree, their singularities, their irreducible factors, and the multiplicity of their points. We also discussed how it was more sensible to study curves in $\mathbb{P}^2_{\mathbb{C}}$, so we defined projective curves (algebraic curves in $\mathbb{P}^2_{\mathbb{C}}$) and redefined the concepts we had introduced for affine curves. In $\mathbb{P}^2_{\mathbb{C}}$, we introduced the resultant and the intersection multiplicity to study the intersection of two projective curves and prove Bézout's theorem, a formula to determine the number of points of intersection between two projective curves up to multiplicity. Moreover, we used the machinery we developed to prove Bézout's theorem to find an equivalence between irreducible conics and cubics under projective coordinates.

In the second half of the project we focused on plane curves. We studied their topological structure and proved the degree-genus theorem, a formula that relates the genus of an orientable surface with the degree of a plane curve. Then, we defined Riemann surfaces and studied functions and differentials on them reaching Riemann-Roch's theorem, a formula that relates the degree of a plane curve with its genus and the dimension of a divisor and a canonical divisor on the Riemann surface defined by the plane curve.

We could keep studying algebraic curves by following the modern algebraic geometry approach: detach them from $\mathbb{P}^2_{\mathbb{C}}$ and study them for an arbitrary algebraic field. Furthermore, we could take a closer look at the singularities of projective curves and, instead of just focusing on plane curves as we did in the last two chapters, attempt to study both the topological structure and the Riemann surfaces defined by singular projective curves.

Chapter 7

Impact analysis

The results achieved in this end-of-degree project have been known for years; I have not discovered anything new. Thus, in this section I am going to talk about the impact that working on this project has had on me.

This project has been my first attempt at studying a branch of mathematics by myself and producing a formal, self-contained article that can be read and understood by others. It has been my first contact with the mathematical world outside of a classroom, and I have enjoyed it a lot. I chose algebraic curves to get a flavor of what pursuing a career in theoretical mathematics would be like and to gather the mathematical machinery I have developed in my courses for the past four years to reach a concrete goal, and, after many hours, I have achieved it.

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